On Order Relations Between Lower Bounds on the MSE of Unbiased Estimators

Koby Todros and Joseph Tabrikian
Dept. of Electrical and Computer Engineering
Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel
Email: todros@bgu.ac.il, joseph@ee.bgu.ac.il

Abstract

Recently, some general classes of non-Bayesian, Bayesian and Hybrid lower bounds on the mean square error (MSE) of estimators have been developed via projection of each entry of the vector of estimation error on some Hilbert subspaces of $L_2$. In this paper, we utilize this framework for derivation of order relations between lower bounds on the MSE of unbiased estimators. We show that some existing and new order relations can be simply obtained by comparing the corresponding Hilbert subspaces on which each entry of the vector of estimation error is projected.

Index Terms

Parameter estimation, mean-square-error bounds, non-Bayesian estimation, uniformly unbiased estimators, performance lower bounds.

I. INTRODUCTION

In [1]-[5] some general classes of non-Bayesian, Bayesian and Hybrid lower bounds on the mean square error (MSE) of estimators were developed via projection of each entry of the vector of estimation error on some Hilbert subspaces of $L_2$. In the non-Bayesian case, these Hilbert subspaces were constructed by applying integral transforms on the likelihood-ratio (LR) function. In the Bayesian and Hybrid cases, the integral transforms were applied on the functions used for computation of bounds in the Weiss-Weinstein family, and on the centered LR function, respectively. By modifying the kernel function of the integral transform, the Hilbert subspaces were modified and a variety of new bounds, as well as existing bounds, were obtained.
In this paper, the framework described above is utilized for derivation of order relations between non-Bayesian bounds for unbiased estimators. We show that order relations between lower bounds on the MSE of unbiased estimators can be simply obtained by comparing the corresponding Hilbert subspaces of $L_2$, on which each entry of the vector of estimation error is projected. In [4] it is shown that some lower bounds on the MSE of unbiased estimators, such as the Cramér-Rao bound (CRB) [6], [7], Bhattacharyya bound (BHB) [8], McAulay-Seedman (MS) [9], Hammersley-Chapman-Robbins (HCR) [10], McAulay-Hofstetter (MH) [11], Abel [12] and QCL [13] bounds, constitute the limits of convergent sequences of lower bounds, which are obtained from the class in [4] via specific choices of sequences of kernel functions. In the paper, we show that order relations between these bounds can be obtained by comparing the limits of the Hilbert subspace sequences corresponding to the chosen sequences of kernel functions.

In [1], [4] a new lower bound, named here as the Cramer-Rao-Fourier (CRF) bound, was derived from the proposed class using the kernel function of the Fourier transform. In cases where the LR function is sampled at $N$ equally spaced test-points in the parameter space, it was shown in [4] that the proposed bound is computed by evaluating the discrete-Fourier-transform (DFT) of the sampled LR function at $J < N$ frequency test-bins. In the paper, we show that in cases where the $L_2$-norm of the spectrum of the sampled LR function is concentrated at these frequency test-bins, the proposed bound is as tight as the MH bound, which is computed using $N > J$ test-points in the parameter space. Order relation between the CRF and Abel bounds is given as well.

The paper is organized as follows. In Section II an overview of the class of lower bounds on the MSE of unbiased estimators, derived in [1], [4] is given. In Section III, some order relations between lower bounds, obtained from the proposed class, are derived. Section IV summarizes the main points of this contribution.

II. A NEW CLASS OF LOWER BOUNDS ON THE MEAN SQUARE ERROR OF UNBIASED ESTIMATORS: A REVIEW

In this section, an overview of the class of lower bounds on the MSE of unbiased estimators, derived in [1], [4] is given. Let $\Theta \subseteq \mathbb{R}^M$ denote a measurable parameter space with finite Lebesgue measure, and consider the estimation of $g (\theta_t)$, where $g : \Theta \to \mathbb{R}^L$ is bounded and deterministic known and $\theta_t \in \Theta$ is deterministic unknown. Moreover, let $\hat{g}(x) \in L^2(X)$ denote an estimator of $g (\theta_t)$, where $x \in X$ is an observation point, $X$ is $P_\theta$-measurable observation space, $P_\theta$ is a family of probability measures on $X$, parameterized by $\theta$, which are absolutely continuous with respect to (w.r.t.) the $\sigma$-finite positive measure $\mu$ on $X$, such that the Radon-Nikodym derivative $f(x; \theta) \triangleq \frac{dP_\theta(x)}{d\mu(x)}$, exists $\forall \theta \in \Theta$, $L_2(X)$ is
the Hilbert space of absolutely square integrable functions, $\zeta: \mathcal{X} \rightarrow \mathbb{C}$, w.r.t. $\mathcal{P}_\theta$, such that the inner product of two elements, $\zeta(x)$, $\zeta'(x)$, in $L_2(\mathcal{X})$ is defined by

$$\langle \zeta(x), \zeta'(x) \rangle_{L_2(\mathcal{X})} \triangleq E_{\zeta;\theta} \left[ \zeta(x) \zeta'(x) \right],$$  \hspace{1cm} (1)

with $(\cdot)^*$ and $E_{\zeta;\theta} [\cdot]$ denoting the complex-conjugate and the expectation w.r.t. $f(x;\theta_t)$, respectively, and $L_2(\mathcal{X}) \triangleq L_2(\mathcal{X}) \times \cdots \times L_2(\mathcal{X})$.

Assuming that $\hat{g}(x)$ is uniformly unbiased, i.e. $E_{\zeta;\theta} [\hat{g}(x)] = g(\theta) \; \forall \theta \in \Theta$, it was shown in [1], [4] that $\hat{g}(x)$-independent lower bounds on the MSE of $\hat{g}(x)$ at $\theta_t$ can be obtained via projection of each entry of the vector of estimation error on any closed subspace of $H_\phi \subset L_2(\mathcal{X})$, where $H_\phi$ contains linear functionals of the LR function. More explicitly, it was shown that

$$\text{MSE} (\hat{g}(x)) \geq C_{H_\phi}, \quad \forall H_\varphi \subset H_\phi \subset L_2(\mathcal{X}),$$  \hspace{1cm} (2)

where $\text{MSE} (\hat{g}(x)) \triangleq E_{\zeta;\theta} \left[ [e(\hat{g}(x)) e^H(\hat{g}(x))] \right]$, is the MSE matrix at $\theta_t$, $e(\hat{g}(x)) \triangleq \hat{g}(x) - g(\theta_t)$ is the vector of estimation error, and

$$C_{H_\phi} \triangleq E_{\zeta;\theta} \left[ p_J (e(\hat{g}(x)) | H_\varphi) p_{J^H} (e(\hat{g}(x)) | H_\phi) \right].$$  \hspace{1cm} (3)

The vector

$$p_J (e(\hat{g}(x)) | H_\varphi) \triangleq \left[ p_J ([e(\hat{g}(x))])_0 | H_\varphi), \ldots, p_J ([e(\hat{g}(x))])_{L-1} | H_\varphi) \right]^T,$$

where $p_J ([e(\hat{g}(x))])_l | H_\varphi)$ is the projection of $[e(\hat{g}(x))])_l$ on $H_\varphi \subset H_\phi \subset L_2(\mathcal{X})$, such that $H_\varphi$ is closed. The space

$$H_\phi \triangleq \left\{ \phi_q(x) \triangleq \int_\Theta q(\theta) \nu(x, \theta) d\theta : q(\theta) \in L_1(\Theta) \right\},$$  \hspace{1cm} (5)

where $q: \Theta \rightarrow \mathbb{C}$, $L_1(\Theta)$ denotes the space of absolutely integrable functions in $\Theta$,

$$\nu(x, \theta) \triangleq \begin{cases} \frac{f(x;\theta)}{f(x;\theta_t)} = \frac{dp_{\theta}(x)}{dp_{\theta_t}(x)}, & f(x;\theta_t) > 0, \\ 0, & \text{otherwise} \end{cases}$$  \hspace{1cm} (6)

denotes the LR function, and it is assumed that $\nu(x, \theta) \in L_2(\mathcal{X}) \quad \forall \theta \in \Theta$. Under this assumption, it was shown in [4] that $H_\phi \subset L_2(\mathcal{X})$. By setting $H_\varphi = H_\phi$ it was shown in [4] that $C_{H_\phi}$ is the integral form of the Barankin bound [14], which is the tightest lower bound on the MSE of uniformly unbiased estimators.
Hence, using the result in (2) a new class of lower bounds on the MSE of uniformly unbiased estimators was derived in [1], [4] via projection of each entry of \( e(\hat{g}(x)) \) on a closed subspace of \( \mathcal{H}_\phi \), which contains linear functionals of elements in the domain of an integral transform defined on the LR function. A general form of this class is given by

\[
C_{\mathcal{H}_\phi^{(h)}} \triangleq \mathbb{E}_{x;\theta_1} \left[ p_f \left( e(\hat{g}(x)) | \mathcal{H}_\phi^{(h)} \right) p^H_f \left( e(\hat{g}(x)) | \mathcal{H}_\phi^{(h)} \right) \right],
\]

where

\[
\mathcal{H}_\phi^{(h)} \triangleq \left\{ \varphi_{a,h}(x) \triangleq \int_\Lambda a^H(\tau) \eta_h(x,\tau) d\tau : a(\tau) \in L^1_\nu(\Lambda) \right\},
\]

and \( a : \Lambda \rightarrow \mathbb{C}^P \). The vector function

\[
\eta_h(x,\tau) \triangleq (T_h(\nu))(x,\tau) = \int_\Theta h(\tau,\theta) \nu(x,\theta) d\theta
\]

is an integral transform on \( \nu(x,\theta) \), where \( h : \Lambda \times \Theta \rightarrow \mathbb{C}^P \) is the kernel of \( T_h \) and \( \Lambda \subseteq \mathbb{R}^M \) is a measurable space with finite Lebesgue-measure. Under some mild assumptions on \( h(\cdot,\cdot) \) and under the assumption that \( \mathcal{H}_\phi^{(h)} \) is closed, it was shown in [4] that \( \mathcal{H}_\phi^{(h)} \) is a closed subspace of \( \mathcal{H}_\phi \). A closed form expression and Geometric interpretation of \( C_{\mathcal{H}_\phi^{(h)}} \) are given in [1], [4].

The bound in (7) constitutes a new family of lower bounds on the MSE of any uniformly unbiased estimator. By modifying \( h(\cdot,\cdot) \), the subspace \( \mathcal{H}_\phi^{(h)} \) is modified and a variety of bounds can be obtained from the proposed class. Since \( \mathcal{H}_\phi \supseteq \mathcal{H}_\phi^{(h)} \), it is concluded from Theorem 1 in subsection III-A that \( C_{\mathcal{H}_\phi} \geq C_{\mathcal{H}_\phi^{(h)}} \). In [4], it is shown that for any invertible integral transform, \( T_h \), which preserves the information in \( \nu(x,\theta) \), \( C_{\mathcal{H}_\phi} = C_{\mathcal{H}_\phi^{(h)}} \). Non-invertible integral transforms may produce less tighter bounds than \( C_{\mathcal{H}_\phi} \). However, as discussed in [1], [4], in many cases, \( C_{\mathcal{H}_\phi} \) is practically incomputable. Therefore, in order to obtain computationally manageable and tight bounds we look for kernels, which yield non-invertible integral transforms, such that the corresponding bound is computable and the significant information in \( \nu(x,\theta) \) is compressed into few elements in the domain of the integral transform.

### III. ORDER RELATION BETWEEN LOWER BOUNDS FOR UNBIASED ESTIMATORS

In [1], [4] it was shown that some lower bounds on the MSE of unbiased estimators constitute the limits of convergent sequences of lower bounds, which are obtained from (7) via specific choices of sequences of kernel functions in \( L^1_\nu(\Theta) \). In this section, we show that order relations between these bounds can be obtained by comparing the limits of the Hilbert subspace sequences corresponding to the chosen sequences of kernel functions.
A. Mathematical background

In this subsection, the mathematical tools needed in derivation of order relations between some lower bounds on the MSE of unbiased estimators are derived. We begin by stating the following theorem.

Theorem 1: Let $\mathcal{H}_\varphi$ and $\mathcal{H}'_{\varphi}$ denote closed subspaces of $\mathcal{H}_\phi$. If $\mathcal{H}_\varphi \supset \mathcal{H}'_{\varphi}$ then $C_{\mathcal{H}_\varphi} \succeq C_{\mathcal{H}'_{\varphi}}$.

Proof: Let

$$u(e(\hat{g}(x))|\mathcal{H}_\varphi) \triangleq e(\hat{g}(x)) - p_{\mathcal{H}}(e(\hat{g}(x))|\mathcal{H}_\varphi),$$

(10)

and

$$u(e(\hat{g}(x))|\mathcal{H}'_{\varphi}) \triangleq e(\hat{g}(x)) - p_{\mathcal{H}}(e(\hat{g}(x))|\mathcal{H}'_{\varphi}),$$

(11)

denote the projection-errors for $\mathcal{H}_\varphi$ and $\mathcal{H}'_{\varphi}$, respectively, where $p_{\mathcal{H}}(e(\hat{g}(x))|\mathcal{H}_\varphi)$ is defined in (4). According to (10) and (11)

$$p_{\mathcal{H}}(e(\hat{g}(x))|\mathcal{H}_\varphi) = p_{\mathcal{H}}(e(\hat{g}(x))|\mathcal{H}'_{\varphi}) + (u(e(\hat{g}(x))|\mathcal{H}'_{\varphi}) - u(e(\hat{g}(x))|\mathcal{H}_\varphi)).$$

(12)

By the Hilbert projection theorem [15] it is implied that

$$[u(e(\hat{g}(x))|\mathcal{H}_\varphi)]_l \perp \mathcal{H}_\varphi, \forall l = 0, \ldots, L - 1.$$  

(13)

and

$$[u(e(\hat{g}(x))|\mathcal{H}'_{\varphi})]_l \perp \mathcal{H}'_{\varphi}, \forall l = 0, \ldots, L - 1.$$  

(14)

Therefore, since $\mathcal{H}_\varphi \supset \mathcal{H}'_{\varphi}$ it is implied by (13) that

$$[u(e(\hat{g}(x))|\mathcal{H}_\varphi)]_l \perp \mathcal{H}'_{\varphi}, \forall l = 0, \ldots, L - 1.$$  

(15)

Thus, due to the fact that $[p_{\mathcal{H}}(e(\hat{g}(x))|\mathcal{H}'_{\varphi})]_l \in \mathcal{H}'_{\varphi}, \forall l = 0, \ldots, L - 1$, then by (14) and (15)

$$r^H(u(e(\hat{g}(x))|\mathcal{H}'_{\varphi}) - u(e(\hat{g}(x))|\mathcal{H}_\varphi)) \perp r^H p_{\mathcal{H}}(e(\hat{g}(x))|\mathcal{H}'_{\varphi}), \forall r \in \mathbb{C}^L.$$  

(16)

Hence, according to (12) and the Pythagorean theorem [16],

$$\left\| r^H p_{\mathcal{H}}(e(\hat{g}(x))|\mathcal{H}_\varphi) \right\|_{L^2(X)}^2 = \left\| r^H p_{\mathcal{H}}(e(\hat{g}(x))|\mathcal{H}'_{\varphi}) \right\|_{L^2(X)}^2,$$

(17)

$$+ \left\| r^H (u(e(\hat{g}(x))|\mathcal{H}'_{\varphi}) - u(e(\hat{g}(x))|\mathcal{H}_\varphi)) \right\|_{L^2(X)}^2.$$

Therefore, due to the fact that $0 \leq \left\| r^H (u(e(\hat{g}(x))|\mathcal{H}'_{\varphi}) - u(e(\hat{g}(x))|\mathcal{H}_\varphi)) \right\|_{L^2(X)}^2 < \infty$, then

$$\left\| r^H p_{\mathcal{H}}(e(\hat{g}(x))|\mathcal{H}_\varphi) \right\|_{L^2(X)}^2 \geq \left\| r^H p_{\mathcal{H}}(e(\hat{g}(x))|\mathcal{H}'_{\varphi}) \right\|_{L^2(X)}^2, \forall r \in \mathbb{C}^L.$$  

(18)
We note that the second limit notation in (20) means that
\[ \phi \{ H_C \phi_k \} \]
where the second equality stems from Theorem 4 in Appendix B. Therefore, if
\[ \lim_{k \to \infty} \phi \{ H_C \phi_k \} = \phi \{ H_C \phi \} \]
Thus, since \( C_H \phi \) and \( C_H' \phi \) are Hermitian matrices, it is implied by (19) that \( C_H \phi \succeq C_H' \phi \).

Hence, by Theorem 1 it is concluded that order relations between lower bounds, \( C_{H_{\phi}}(h_1) \) and \( C_{H_{\phi}}(h_2) \), can be obtained by simply comparing the Hilbert subspaces \( H_{\phi}(h_1) \) and \( H_{\phi}(h_2) \). Moreover, in the following theorem it is shown that order relation between any two bounds, \( C_1 \) and \( C_2 \), which constitute the limits of two convergent sequences of bounds, \( \{ C_{H_{\phi}}(h_{1,k}) \} \) and \( \{ C_{H_{\phi}}(h_{2,k}) \} \) can be obtained by comparing the limits of the Hilbert subspace sequences, \( \{ H_{\phi}(h_{1,k}) \} \) and \( \{ H_{\phi}(h_{2,k}) \} \).

**Definition 1:** Let \( \{ h_k(\cdot, \cdot) \}_{k \in \mathbb{N}} \) denote a sequence of kernel functions. Then,
\[
\lim_{k \to \infty} H_{\phi}(h_k) \triangleq \left\{ \begin{array}{ccc}
\varphi_a(x) & \triangleq & \lim_{k \to \infty} \varphi_{a,h_k}(x) \\
: & & \varphi_{a,h_k}(x) \in H_{\phi}(h_k), \ k \in \mathbb{N}
\end{array} \right\}.
\]
We note that the second limit notation in (20) means that \( \varphi_{a,h_k}(x) \to \varphi_a(x) \) for a.e. \( x \in \mathcal{X} \), as \( k \to \infty \).

By theorem 3 in Appendix A it is implied that \( \varphi_a(x) \in H_{\phi} \) and \( \varphi_{a,h_k}(x) \to \varphi_a(x) \) in the \( L_2(\mathcal{X}) \)-norm, as \( k \to \infty \).

**Theorem 2:** Let \( C_n \triangleq \lim_{k \to \infty} C_{H_{\phi}(h_{n,k})} \), \( n = 1, 2 \), where \( \{ h_{n,k}(\cdot, \cdot) \}_{n \in \mathbb{N}} \), \( n = 1, 2, \ k \in \mathbb{N} \), denote two sequences of kernel functions, such that \( \{ H_{\phi}(h_{n,k}) \}, \ n = 1, 2, \ k \in \mathbb{N} \) are two convergent sequences of closed subspaces of \( H_{\phi} \). If \( \lim_{k \to \infty} H_{\phi}(h_{1,k}) \succeq \lim_{k \to \infty} H_{\phi}(h_{2,k}) \), then \( C_1 \succeq C_2 \), where equality holds if \( \lim_{k \to \infty} H_{\phi}(h_{1,k}) = \lim_{k \to \infty} H_{\phi}(h_{2,k}) \).

**Proof:** According to the definitions of \( C_1 \) and \( C_2 \) above,
\[
C_n = \lim_{k \to \infty} C_{H_{\phi}(h_{n,k})} = C_{\lim_{k \to \infty} H_{\phi}(h_{n,k})}, \ n = 1, 2,
\]
where the second equality stems from Theorem 4 in Appendix B. Therefore, if \( \lim_{k \to \infty} H_{\phi}(h_{1,k}) = \lim_{k \to \infty} H_{\phi}(h_{2,k}) \) then \( C_1 = C_2 \). Moreover, if \( \lim_{k \to \infty} H_{\phi}(h_{1,k}) \supset \lim_{k \to \infty} H_{\phi}(h_{2,k}) \), then by Theorem 1 \( C_1 \succeq C_2 \).

**B. Limits of Hilbert subspace sequences**

In this subsection, the limits of the Hilbert subspace sequences corresponding to the kernel sequences, which were used in [4] for derivation of the CRB, BHB, MS, HCR, MH, Abel, CQL, and CRF bounds, are calculated.
1) The Cramér-Rao bound [6], [7]:

In [4] it was shown that the CRB is given by

\[ C_{\text{CRB}} = \lim_{k \to \infty} C_{H^{(v_{\text{CRB}},k)}} \]

where the following sequence of kernel functions was used.

\[ v_{\text{CRB},k}(\theta) = -\psi_k^{(1)}(\theta_t - \theta), \quad k \in \mathbb{N}. \]  \hspace{1cm} (22)

The vector

\[ \psi_k^{(Q)}(\theta) \triangleq \left( \frac{\partial^Q \psi_k(\theta)}{\partial \theta^{\otimes Q}} \right)^T, \quad Q \in \mathbb{N}, \]  \hspace{1cm} (23)

where \( \frac{\partial^Q}{\partial \theta^{\otimes Q}} \) denotes the row vector of derivatives \( \frac{\partial^Q \psi_k(\theta)}{\partial \theta_1 \cdots \partial \theta_i} \), \( i_q = 1, \ldots, M \), and the sequence \( \{\psi_k(\theta)\}_{k \in \mathbb{N}} \triangleq \{ k^M \psi_k(\theta) \}_{k \in \mathbb{N}} \) is comprised of “test-functions”, such that \( \psi : \mathbb{R}^M \to \mathbb{R} \) denotes an infinitely differentiable and compactly supported “test-function” with \( \text{supp} \{\psi(\cdot)\} \subset \Theta \) and \( \int_{\mathbb{R}^M} \psi(\theta) \, d\theta = 1 \). We note that \( \lim_{k \to \infty} \psi_k(\theta) = \delta(\theta) \), where \( \delta(\cdot) \) is the Dirac’s delta function.

According to (8), (9), and (22)

\[ H^{(v_{\text{CRB}},k)}(\varphi) \rightarrow \left\{ \varphi(x) = a^H \int_{\Theta} -\psi_k^{(1)}(\theta_t - \theta) \nu(x, \theta) \, d\theta : a \in \mathbb{C}^M \right\}. \]  \hspace{1cm} (24)

Hence, using the definition in (20) followed by integration by parts and applying the result of theorem (8.15) in [16] regarding the limit of the convolution integral, it can be shown that

\[ H^{(v_{\text{CRB}},k)} \rightarrow \left\{ \varphi(x) = a^H \left. \frac{\partial \nu(x, \theta)}{\partial \theta} \right|_{\theta = \theta_t}^T : a \in \mathbb{C}^M \right\}. \]  \hspace{1cm} (25)

2) The Bhattacharyya bound [8]:

In [4] it was shown that the \( Q \)-th order BHB is given by

\[ C_{\text{BHB}}^{(Q)} = \lim_{k \to \infty} C_{H^{(v_{\text{BHB}},k)}} \]

where

\[ v_{\text{BHB},k}(\theta) = \left[ -\psi_k^{(1)^T}(\theta_t - \theta), \ldots, (-1)^Q \psi_k^{(Q)^T}(\theta_t - \theta) \right]^T, \]  \hspace{1cm} (26)

\( k \in \mathbb{N} \). Hence, in similar to (25) it can be shown that

\[ H^{(v_{\text{BHB}},k)} \rightarrow \left\{ \varphi(x) = \sum_{q=1}^{Q} a_q^H \frac{\partial \nu(x, \theta)}{\partial \theta^{\otimes q}} \bigg|_{\theta = \theta_t}^T : a_q \in \mathbb{C}^{M^q}, \forall q = 1, \ldots, Q \right\}. \]  \hspace{1cm} (27)

3) The McAulay-Seidman bound [9]:

In [4] it was shown that the \( N \)-th order MS bound is given by

\[ C_{\text{MS}} = \lim_{k \to \infty} C_{H^{(v_{\text{MS}},k)}} \]

where

\[ v_{\text{MS},k}(\theta) = \left[ \psi_k(\theta_0 - \theta), \ldots, \psi_k(\theta_{N-1} - \theta) \right]^T, \quad k \in \mathbb{N}, \]  \hspace{1cm} (28)
and $\theta_n \in \Theta$, $n = 0, \ldots, N - 1$, denote test-points. Hence, according to (8), (9), and (28)
\[
\mathcal{H}_{\varphi}^{(v^{(N)}_{MS,k})} = \left\{ \varphi(x) = \sum_{n=0}^{N-1} a_n^* \int_{\Theta} \psi_k(\theta_n - \theta) \nu(x, \theta) d\theta \right\}
\]
\[
: a_n \in \mathbb{C}, \forall n = 0, \ldots, N - 1 \right\}. \tag{29}
\]

Therefore, using the definition in (20) and applying the result of theorem (8.15) in [16] regarding the limit of the convolution integral, it can be shown that
\[
\mathcal{H}_{\varphi}^{(v^{(N)}_{MS,k})} \rightarrow \left\{ \varphi(x) = \sum_{n=0}^{N-1} \int_{\Theta} \psi_k(\theta_n - \theta) \nu(x, \theta) d\theta \right\}
\]
\[
: a_n \in \mathbb{C}, \forall n = 0, \ldots, N - 1 \right\}. \tag{30}
\]

4) The Hammersley-Chapman-Robbins bound [10]:

In [4] it was shown that the $N$-th order HCR bound is given by $C_{HCR}^{(N)} = \lim_{k \to \infty} \mathcal{C}_{\mathcal{H}_{\varphi}^{(v^{(N)}_{HCR,k})}}$, where
\[
v^{(N)}_{HCR,k}(\theta) = [\psi_k(\theta_0 - \theta), \ldots, \psi_k(\theta_{N-1} - \theta)]^T, \tag{31}
\]
k $\in \mathbb{N}$. Hence, in similar to (30) it can be shown that
\[
\mathcal{H}_{\varphi}^{(v^{(N)}_{HCR,k})} \rightarrow \left\{ \varphi(x) = \sum_{n=0}^{N-1} \psi_k(\theta_n - \theta_n) + a_n^* \nu(x, \theta_n) \right\}
\]
\[
: a_n \in \mathbb{C}, \forall n = 0, \ldots, N \right\}. \tag{32}
\]

5) The McAulay-Hofstetter bound [11]:

In [4] it was shown that the $N$-th order MH bound is given by $C_{MH}^{(N)} = \lim_{k \to \infty} \mathcal{C}_{\mathcal{H}_{\varphi}^{(v^{(N)}_{MH,k})}}$, where
\[
v^{(N)}_{MH,k}(\theta) = [v_{CRB,k}(\theta), v_{HCR,k}(\theta)]^T, \tag{33}
\]
k $\in \mathbb{N}$. Hence, in similar to (25) and (30) it can be shown that
\[
\mathcal{H}_{\varphi}^{(v^{(N)}_{MH,k})} \rightarrow \left\{ \varphi(x) = a^H \left. \frac{\partial \nu(x, \theta)}{\partial \theta} \right|_{\theta=\theta_n} + \sum_{n=0}^{N-1} b_n^* \nu(x, \theta_n) \right\}
\]
\[
: a \in \mathbb{C}^M, b_n \in \mathbb{C}, \forall n = 0, \ldots, N - 1 \right\}. \tag{34}
\]

6) The Abel bound [12]:

In [4] it was shown that the $(Q,N)$-th order Abel bound is given by $C_{Abel}^{(Q,N)} = \lim_{k \to \infty} \mathcal{C}_{\mathcal{H}_{\varphi}^{(v^{(Q,N)}_{Abel,k})}}$, where
\[
v^{(Q,N)}_{Abel,k}(\theta) = [v_{BHB,k}(\theta), v_{HCR,k}(\theta)]^T, \tag{35}
\]
k $\in \mathbb{N}$.
Hence, in similar to (27) and (34) it can be shown that

\[
\mathcal{H}_\varphi^{(v_{Q,N})} \to \varphi(x) = \sum_{q=1}^{Q} a_q H_q^T \frac{\partial^q \nu(x, \theta)}{\partial \theta^q} \bigg|_{\theta = \theta_n}^{T} + \sum_{n=0}^{N-1} b_n^x \nu(x, \theta_n) + b_N^x \\
: a_q \in \mathbb{C}^{M_q}, \forall q = 1, \ldots, Q \\
, b_n \in \mathbb{C}, \forall n = 0, \ldots, N \}
\] (36)

7) The QCL bound [13]:

In [4] it was shown that the \((Q, N)\)-th order QCL bound is given by \(C_{\text{QCL}}^{(Q,N)} = \lim_{k\to\infty} \mathcal{C}_{\mathcal{H}_\varphi^x}^{(v_{Q,N})} \), where \(Q \triangleq [Q_0, \ldots, Q_{N-1}]\) denotes a vector of derivative orders, and the following sequence of kernel functions was used.

\[
v_{QCL, k}^{(Q,N)}(\theta) = \left[ v_{0,k}^{T}(\theta), \ldots, v_{N-1,k}^{T}(\theta) ; v_{MS,k}^{(N)T}(\theta) \right]^T, \]

such that

\[
v_{n,k}^{(1)}(\theta_n - \theta), \ldots, (-1)^{Q_n} \psi_{k}^{(Q_n)T}(\theta_n - \theta) \]

Hence, in similar to (27) and (30) it can be shown that

\[
\mathcal{H}_\varphi^{(v_{QCL, k})} \to \varphi(x) = \sum_{n=0}^{N-1} \sum_{q=1}^{Q_n} a_{n,q} H_q^T \frac{\partial^q \nu(x, \theta)}{\partial \theta^q} \bigg|_{\theta = \theta_n}^{T} + \sum_{n=0}^{N-1} b_n^x \nu(x, \theta_n) \\
: a_{n,q} \in \mathbb{C}^{M_q}, \forall q = 1, \ldots, Q_n \\
, b_n \in \mathbb{C}, \forall n = 0, \ldots, N - 1 \}
\] (39)

8) The CRF bound [1], [4]:

In [4] it was shown that the \((J, N)\)-order CRF bound is given by \(C_{\text{CRF}}^{(J,N)} = \lim_{k\to\infty} \mathcal{C}_{\mathcal{H}_\varphi^x}^{(v_{CRF, k})} \), where \(v_{CRF, k}^{(J,N)}(\theta) = \left[ v_{CRB,k}^{T}(\theta), v_{MS,k}^{(N)T}(\theta) \right]^T, k \in \mathbb{N} \) and \(v_{MS,k}^{(N)}(\theta)\), defined in (28), contains \(N\) equally spaced test-points in \(\Theta\). The discrete-Fourier-transform (DFT) matrix with \(J < N\) orthogonal rows is denoted by \(W \in \mathbb{C}^{J \times N}\), where

\[
[W]_{j,n} = \exp(-i\Omega_j^T \theta_n), j \in J, n = 0, \ldots, N - 1,
\] (41)
\(\Omega_j, j \in J\) are frequency test-bins, and \(J \subset \{0, \ldots, N - 1\}\) is an index set of the selected frequency test-bins with cardinality \(|J| = J < N\). Hence, in similar to (25) and (30) it can be shown that
\[
H_{\varphi}^{(\mathcal{V}^{(j,N)}_{\text{CRF,}k})} \rightarrow \left\{ \varphi(x) = a^H \frac{\partial \nu(x, \theta)}{\partial \theta}\right|_{\theta = \theta_t}^T + \sum_{j \in J} b_j^* \hat{\nu}(x, \Omega_j) : a \in \mathbb{C}^M, b_j \in \mathbb{C}, \forall j \in J \right\}, \tag{42}
\]
as \(k \to \infty\), where
\[
\hat{\nu}(x, \Omega_j) \triangleq N^{-1} \sum_{n=0}^{N-1} \nu(x, \theta_n) \exp(-i\Omega_j^T \theta_n) \tag{43}
\]
denotes the DFT of the sequence \(\{\nu(x, \theta_n)\}_{n=0}^{N-1}\), evaluated at \(\Omega_j, j \in J\).

### C. Order relations

In this subsection, we use the result of Theorem 2 and the limits of the Hilbert subspace sequences calculated in the previous subsection, in order to prove some order relations.

1) **CRB vs. BHB:**

**Proposition 1:**
\[
C_{\text{BHB}}^{(Q)} \succeq C_{\text{CRB}}, \tag{44}
\]
where equality holds if \(Q = 1\) in (26).

**Proof:** According to (25) and (27), \(\lim_{k \to \infty} H_{\varphi}^{(\mathcal{V}^{(Q)}_{\text{BHB,}k})} \supseteq \lim_{k \to \infty} H_{\varphi}^{(\mathcal{V}_{\text{CRB,}k})}\), where equality holds for \(Q = 1\). Therefore, by theorem 2 (44) holds. \(\blacksquare\)

2) **MS vs. HCR:**

**Proposition 2:**
\[
C_{\text{HCR}}^{(N)} \succeq C_{\text{MS}}^{(N)}, \tag{45}
\]
where equality holds if \(\theta_0 = \theta_t\) in (28) and (31).

**Proof:** According to (30) and (32), \(\lim_{k \to \infty} H_{\varphi}^{(\mathcal{V}^{(N)}_{\text{HCR,}k})} \supseteq \lim_{k \to \infty} H_{\varphi}^{(\mathcal{V}^{(N)}_{\text{MS,}k})}\), where equality holds if \(\theta_0 = \theta_t\). Therefore, by theorem 2 it is concluded that (45) holds. \(\blacksquare\)

3) **MS vs. CRB:**

In the following proposition, a sufficient condition according to which the MS bound is tighter than the CRB is derived.

**Proposition 3:** Let
\[
C_{\text{MS,} \epsilon}^{(N)} \triangleq \lim_{k \to \infty} C_{\mathcal{H}_{\varphi}^{(\mathcal{V}^{(N)}_{\text{MS,}k})}} \tag{46}
\]
where \( v_{MS,\varepsilon,k}^{(N)}(\theta) = v_{MS,k}^{(N)}(\theta) \) with the following set of test-points:

\[
\theta_n = \begin{cases} 
\theta_t + \varepsilon u_n, & n = 0, \ldots, M - 1 \\
\theta_t, & n = M, \ldots, 2M - 1,
\end{cases}
\]  
(47)

\[
[u_n]_m \triangleq \begin{cases} 
1, & m = n + 1 \\
0, & \text{otherwise}
\end{cases}, \quad m = 1, \ldots, M,
\]  
(48)

where \( \varepsilon > 0 \) denotes a constant, \( N \) denotes the number of test-points, \( M \) is the dimension of \( \Theta \), and it is assumed that \( N > 2M \). Then,

\[
\lim_{\varepsilon \to 0} C_{MS,\varepsilon}^{(N)} \geq C_{CRB}.
\]  
(49)

**Proof:** Using (30) and (47), it can be shown that

\[
\lim_{k \to \infty} \mathcal{H}_{\nu_{MS,\varepsilon,k}}^{(N)} \supset \mathcal{H}_{\varepsilon}(\varepsilon) \triangleq \mathcal{H}_{\varepsilon}^{(\varepsilon)}.
\]  
(50)

Therefore, since \( \lim_{k \to \infty} \mathcal{H}_{\nu_{MS,\varepsilon,k}}^{(N)} \supset \mathcal{H}_{\varepsilon}(\varepsilon), \forall \varepsilon > 0 \), it is concluded that

\[
\lim_{\varepsilon \to 0} \lim_{k \to \infty} \mathcal{H}_{\nu_{MS,k}}^{(N)} \supset \lim_{\varepsilon \to 0} \mathcal{H}_{\varepsilon}(\varepsilon).
\]  
(51)

According to (25) and (50)

\[
\lim_{\varepsilon \to 0} \mathcal{H}_{\nu_{CRB,k}}^{(\varepsilon)} = \left\{ \varphi(x) = \sum_{n=0}^{M-1} \alpha_n^* \lim_{\varepsilon \to 0} \frac{\nu(x, \theta_t + \varepsilon u_n) - \nu(x, \theta_t)}{\varepsilon} : \alpha_n \in \mathbb{C} \forall n = 0, \ldots, M - 1 \right\}
\]  
(52)

\[
= \left\{ \varphi(x) = \alpha^H \frac{\partial \nu(x, \theta)}{\partial \theta} \Big|_{\theta = \theta_t}^T : \alpha \in \mathbb{C}^M \right\}
\]

\[
= \lim_{k \to \infty} \mathcal{H}_{\nu_{CRB,k}}^{(\varepsilon)}.
\]
where the limit w.r.t. $ε$ exists under the assumption that the first-order derivative of $f(x; θ)$ w.r.t. each entry of $θ$ exists and defines a continuous function on $Θ$. Therefore, substitution of (52) into (51) yields

$$\lim_{ε \to 0} \lim_{k \to ∞} \mathcal{H}_φ(v_{MS,k,x}^{(N)}) ≥ \lim_{k \to ∞} \mathcal{H}_φ(v_{CRB,k}^{(N)})$$

(53)

Hence, according to theorem 2 it is implied that (49) holds.

In conclusion, $C_{MS}^{(N)} ≥ C_{CRB}$ when the test-points, $θ_n$, $n = 0, \ldots, N − 1$, are chosen as in (47) and $ε → 0$.

4) QCL vs. Abel, MH, MS and HCR:

**Proposition 4:** If $θ_0 = θ_t$ in (28), (31), (33), (35), and (37), and if $Q_0 ≥ Q$ in (37), then

$$C_{QCL}^{(Q,N)} ≥ C_{Abel}^{(Q,N)} ≥ C_{MH}^{(N)} ≥ C_{MS}^{(N)} = C_{HCR}^{(N)}.$$  

(54)

**Proof:** According to (30), (32), (34), (36) and (39), one can notice that if $θ_0 = θ_t$ and $Q_0 ≥ Q$, then

$$\lim_{k \to ∞} \mathcal{H}_φ(v_{QCL,k}^{(Q,N)}) ≥ \lim_{k \to ∞} \mathcal{H}_φ(v_{Abel,k}^{(Q,N)}) ≥ \lim_{k \to ∞} \mathcal{H}_φ(v_{MH,k}^{(N)}) ≥ \lim_{k \to ∞} \mathcal{H}_φ(v_{MS,k}^{(N)}) = \lim_{k \to ∞} \mathcal{H}_φ(v_{HCR,k}^{(N)}).$$

(55)

Therefore, by Theorem 2, (54) holds.

5) CRF vs. MH:

In the following proposition, a sufficient condition for equality between $C_{CRF}^{(J,N)}$ and $C_{MH}^{(N)}$ is derived.

**Proposition 5:** If $\|\hat{ν}(x, Ω_j)\|_{L_2(X)} = 0 \ \forall j \in \{0, \ldots, N − 1\} \setminus J$, where $\hat{ν}(x, Ω_j)$ is defined in (43) and $J \subset \{0, \ldots, N − 1\}$ is an index set of the selected frequency test-bins with cardinality $|J| = J < N$, then

$$C_{CRF}^{(J,N)} = C_{MH}^{(N)}.$$  

(56)

**Proof:** In order to prove this proposition, we show that $\lim_{k \to ∞} \mathcal{H}_φ(v_{CRF,k}^{(J,N)}) = \lim_{k \to ∞} \mathcal{H}_φ(v_{MH,k}^{(N)})$. Since the DFT matrix $W$ in (41) is comprised of $J$ orthogonal rows, then any sequence $\{b_j\}_{j \in J} \subset C^J$ can be uniquely represented as

$$b_j = \frac{1}{N} \sum_{m=0}^{N−1} α_m \exp \left(-iΩ_j^T θ_m \right), \ j \in J,$$

(57)

where $α_m \in C \ \forall m = 0, \ldots, N − 1$. Moreover, since $\|\hat{ν}(x, Ω_j)\|_{L_2(X)} = 0 \ \forall j \in \{0, \ldots, N − 1\} \setminus J$, then by the norm properties [16] it is implied that $\hat{ν}(x, Ω_j) = 0$ for a.e. $x \in X$ and $∀ j ∈ J$. 

March 7, 2010 DRAFT
\{0, \ldots, N - 1\} \setminus J$. Therefore, using the definition of the inverse DFT it is implied that

\[ \nu(x, \theta_m) = \frac{1}{N} \sum_{j \in J} \hat{\nu}(x, \Omega_j) \exp(i\Omega_j^T \theta_m), \ m = 0, \ldots, N - 1, \tag{58} \]

for a.e. $x \in \mathcal{X}$. Hence, using (34), (42), (43), (57), and (58), it can be easily verified that $\varphi(x) \in \lim_{k \to \infty} \mathcal{H}_{\varphi}(\hat{\nu}_{\text{CRF},k}) \Rightarrow \varphi(x) \in \lim_{k \to \infty} \mathcal{H}_{\varphi}(\hat{\nu}_{\text{MH},k}),$ and $\varphi(x) \in \lim_{k \to \infty} \mathcal{H}_{\varphi}(\hat{\nu}_{\text{MH},k}) \Rightarrow \varphi(x) \in \lim_{k \to \infty} \mathcal{H}_{\varphi}(\hat{\nu}_{\text{CRF},k}).$

Thus, it is concluded that $\lim_{k \to \infty} \mathcal{H}_{\varphi}(\hat{\nu}_{\text{CRF},k}) = \lim_{k \to \infty} \mathcal{H}_{\varphi}(\hat{\nu}_{\text{MH},k}),$ and hence, by Theorem 2, $C_{\text{CRF}}^{(J,N)} = C_{\text{MH}}^{(N)}$. \hfill \qed

In conclusion, if the condition in Proposition 5 is satisfied, then $C_{\text{CRF}}^{(J,N)}$, which is computed with $J < N$ frequency test-bins in the frequency domain $\Lambda$, is as tight as $C_{\text{MH}}^{(N)}$, which is computed with $N$ test-points in the parameter space $\Theta$. Therefore, in this case the CRF bound is preferable upon the MH bound. Moreover, since $C_{\text{MH}}^{(N)} \geq C_{\text{MH}}^{(N')}$ $\forall N > N'$, then under the condition of Proposition 5 it is concluded that $C_{\text{CRF}}^{(J,N)} \geq C_{\text{MH}}^{(N')}$ $\forall N > N'$. In a similar manner, it can be shown that if $\theta_0 = \theta_t$ in (35) and (40), and if the condition in Proposition 5 is satisfied, then $C_{\text{CRF}}^{(J,N)} \geq C_{\text{Abel}}^{(1,N')}$ $\forall N > N'$, where equality holds for $N = N'$.

IV. DISCUSSION

In this paper, a new approach for derivation of order relations between non-Bayesian lower bounds on the MSE of unbiased estimators is presented. Using this approach, it was shown that some existing and new order relations can be simply obtained by comparing some Hilbert subspaces of $L_2$. Moreover, a sufficient condition, according to which the CRF bound in [1], [4] is preferable upon the MH and Abel bounds is derived. Finally, we note that this approach can be used for derivation of more order relations, and can be extended for the Bayesian and Hybrid cases as well.

APPENDIX A

**Theorem 3:** let $\{\phi_k(x)\}$ denote a sequence in $\mathcal{H}_\phi$, where $\mathcal{H}_\phi$ is defined in (5) and constitutes a closed subspace of $L_2(\mathcal{X})$. If $\phi_k(x) \to \phi(x)$ for a.e. $x \in \mathcal{X}$, as $k \to \infty$. Then $\phi(x) \in \mathcal{H}_\phi$ and $\phi_k(x) \to \phi(x)$ in the $L_2(\mathcal{X})$-norm, as $k \to \infty$.

**Proof:** First, we find a function in $L_2(\mathcal{X})$, which dominates each element of $\{\phi_k(x)\}$. According to (5)

\[ \phi_k(x) = \int_\Theta q_k(\theta) \nu(x, \theta) d\theta, \ k \in \mathbb{N}, \tag{59} \]

March 7, 2010 DRAFT
where \( q_k(\theta) \in \mathcal{L}_1(\Theta) \ \forall k \in \mathbb{N} \). Therefore,

\[
|\phi_k(x)| = \left| \int_{\Theta} q_k(\theta) \nu(x, \theta) \, d\theta \right| \leq \int_{\Theta} |q_k(\theta)| \nu(x, \theta) \, d\theta \leq \left( \int_{\Theta} |q_k(\theta)| \, d\theta \right) \max_{\theta \in \Theta} \{\nu(x, \theta)\}, \ \forall x \in \mathcal{X},
\]

(60)

where the first inequality stems from the properties of the Lebesgue integral and from the fact that \( \nu(x, \theta) \in \mathcal{L}^+(\mathcal{X} \times \Theta) \), and the second inequality stems from the fact that \( \nu(x, \theta) \leq \max_{\theta \in \Theta} \{\nu(x, \theta)\} \) \( \forall x \in \mathcal{X} \). Since \( q_k(\theta) \in \mathcal{L}_1(\Theta) \ \forall k \in \mathbb{N} \) then there exists a positive constant \( c \), such that \( \int_{\Theta} |q_k(\theta)| \, d\theta \leq c \) \( \forall k \in \mathbb{N} \). Hence, by (60)

\[
|\phi_k(x)| \leq c \cdot \max_{\theta \in \Theta} \{\nu(x, \theta)\}, \ \forall x \in \mathcal{X}.
\]

(61)

Moreover, let \( \phi_{\text{Dom}}(x) \triangleq c \cdot \max_{\theta \in \Theta} \{\nu(x, \theta)\} \). Since \( \nu(x, \theta) \in \mathcal{L}_2(\mathcal{X}) \ \forall \theta \in \Theta \), then \( \phi_{\text{Dom}}(x) \in \mathcal{L}_2(\mathcal{X}) \) as well. Thus, each element of \( \{\phi_k(x)\} \) is dominated by \( \phi_{\text{Dom}}(x) \in \mathcal{L}_2(\mathcal{X}) \).

Therefore, by the dominated convergence theorem in \( \mathcal{L}_p \) spaces, stated in theorem 5.2.2 in [17], it is concluded that \( \phi(x) \in \mathcal{L}_2(\mathcal{X}) \) and \( \phi_k(x) \to \phi(x) \) in the \( \mathcal{L}_2(\mathcal{X}) \)-norm, as \( k \to \infty \). Since \( \mathcal{H}_\phi \) is closed, the limit of any convergent sequence in \( \mathcal{H}_\phi \) is contained in \( \mathcal{H}_\phi \). Thus, \( \phi(x) \in \mathcal{H}_\phi \).

**APPENDIX B**

**Theorem 4:** Let \( \{C_{\mathcal{H}_{\phi}^{(h_k)}}\} \) denote a convergent sequence of lower bounds, obtained from the class in (7), such that \( \{\mathcal{H}_{\phi}^{(h_k)}\} \) is a convergent sequence of closed subspaces of \( \mathcal{H}_\phi \), obtained from (8). Then,

\[
\lim_{k \to \infty} C_{\mathcal{H}_{\phi}^{(h_k)}} = C_{\lim_{k \to \infty} \mathcal{H}_{\phi}^{(h_k)}}.
\]
Proof:

\[
\lim_{k \to \infty} C \left[ \left. \frac{\mathcal{H}_\varphi^{(h_k)}}{\mathcal{H}_\varphi} \right|_{m,n} \right] = \lim_{k \to \infty} E_{x \theta t} \left[ p_J \left( \left. \frac{e(g(x))}{\mathcal{H}_\varphi^{(h_k)}} \right|_{m} \right) \right] = \lim_{k \to \infty} p_J \left( \left. \frac{e(g(x))}{\mathcal{H}_\varphi^{(h_k)}} \right|_{m} \right) \mathcal{L}_2(X) = \mathcal{L}_2(X) = E_{x \theta t} \left[ p_J \left( \left. \frac{e(g(x))}{\mathcal{H}_\varphi^{(h_k)}} \right|_{m} \right) \mathcal{L}_2(X) = \mathcal{L}_2(X) \right].
\]

The first equality stems from (7), and from the definition of \( p_J (e(g(x)) \cdot \) in (4), the second equality stems from (1) and from proposition (5.21) in [16], regarding the continuity of the inner-product operator, the third equality stems from theorem 5 in Appendix C, the fourth equality stems from (1) and from the definition of \( p_J (e(g(x)) \cdot \) in (4), and the fifth equality stems from (7).

APPENDIX C

Theorem 5: Let \( \{h_k (\cdot, \cdot)\}, k \in \mathbb{N} \), denote a sequence of kernel functions, such that \( \{\mathcal{H}_\varphi^{(h_k)}\} \) is a convergent sequences of closed subspaces of \( \mathcal{H}_\varphi \). Then, \( \forall y(x) \in \mathcal{L}_2(X) \),

\[
\lim_{k \to \infty} p_J \left( y(x) \left| \mathcal{H}_\varphi^{(h_k)} \right. \right) = p_J \left( y(x) \left| \lim_{k \to \infty} \mathcal{H}_\varphi^{(h_k)} \right. \right).
\]

Proof: According to (8), (20), and the Hilbert projection theorem [15], there exist unique

\[
\varphi_{a,h_k} = \left( y(x) \left| \mathcal{H}_\varphi^{(h_k)} \right. \right) \in \mathcal{H}_\varphi^{(h_k)},
\]

and

\[
\varphi_a = \left( y(x) \left| \lim_{k \to \infty} \mathcal{H}_\varphi^{(h_k)} \right. \right) \in \lim_{k \to \infty} \mathcal{H}_\varphi^{(h_k)},
\]

such that

\[
\langle y(x) - \varphi_{a,h_k}, \varphi_{a,h_k} \rangle_{L_2(X)} = 0, \forall a(\tau) \in \mathcal{L}_2^P(A),
\]

and

\[
\langle y(x) - \varphi_a, \varphi_a \rangle_{L_2(X)} = 0, \forall a(\tau) \in \mathcal{L}_2^P(A).
\]
Since (66) holds for any $k \in \mathbb{N}$, then
\[
\lim_{k \to \infty} \langle y(x) - \varphi_{\tilde{a}, h_k}(x), \varphi_{a, h_k}(x) \rangle_{L^2(X)} = 0, \quad \forall (\tau) \in \mathcal{L}^P_1(A) .
\] (68)

Therefore, according to proposition (5.21) in [16], regarding the continuity of the inner-product operator, it is implied that $\forall (\tau) \in \mathcal{L}^P_1(A)$
\[
\langle y(x) - \lim_{k \to \infty} \varphi_{\tilde{a}, h_k}(x), \lim_{k \to \infty} \varphi_{a, h_k}(x) \rangle_{L^2(X)} = 0.
\] (69)

Hence, by the definition of $\varphi_{\tilde{a}}(x)$ in (20) it is concluded that
\[
\langle y(x) - \varphi_{\tilde{a}}(x), \varphi_{a}(x) \rangle_{L^2(X)} = 0, \quad \forall (\tau) \in \mathcal{L}^P_1(A) .
\] (70)

Finally, by comparing (67) and (70) and applying that $\varphi_{\hat{a}}(x)$ in (65) is unique, it is concluded that
\[
p_J \left( y(x) \mid \lim_{k \to \infty} \mathcal{H}_{\varphi}^{(h_k)} \right) = \varphi_{\hat{a}}(x) = \varphi_{\tilde{a}}(x) = \lim_{k \to \infty} p_J \left( y(x) \mid \mathcal{H}_{\varphi}^{(h_k)} \right).
\] (71)

REFERENCES


