A Computer Simulation Study of the Location Estimation Problem in $\nabla^2 \psi = -\phi \psi$

G. A. Tsihrintzis and P. S. Lampropoulo
Department of Informatics, University of Piraeus
80 Karaoli Dimitriou Street, Piraeus 18534, Greece
E-mail: {geoatsi, vlamp}@uniipi.gr

Abstract: We evaluate via computer simulation the performance of the solution that we proposed in [1] for the location estimation problem associated with the partial differential equation $\nabla^2 \psi = -\phi \psi$. Our study evaluates the robustness to noise of the maximum likelihood estimation procedure and its extension to maximum a posteriori estimation.

Recently [1], we addressed an inverse problem associated with the partial differential equation $\nabla^2 \psi = -\phi \psi$ which arises in electrocardiology [2] and low frequency diffraction tomography. We considered the "object function" $\phi$ as a spatially-shifted version of an otherwise completely known object function, i.e., $\phi(r) = \phi_0(r - r_0)$, where $r_0$ is the unknown location of the object. More specifically, we derived a translation property relating the fields scattered by the object $\phi_0$ placed at different locations and used this translation property to derive an algorithm for the computation of the maximum likelihood estimate of the object location from noisy scattered field measurements.

Our approach is similar to that followed for a related problem initially posed within the domain of Computerized [3] and linearized Diffraction [4] Tomography and later extended into the domain of exact (nonlinear) scattering theory [5] and availability of intensity-only data [6]. However, while this previous work concentrated on high frequency wave probing, the present work applies to the regime of static fields. More specifically, we considered the configuration in Fig. 1, in which we defined a fixed Cartesian coordinate system with axes $x_1$ and $x_2$ and a second system that is centred at the origin of the $(x_1; x_2)$ system and has axes $(t; s)$ with unit vectors $u$ and $v$, respectively. The $(t; s)$ system is allowed to rotate around the common origin, with the $t$ axis forming an angle $\theta$ with the positive $x_1$ axis. An object, described by the object function $\phi$ and fully contained in the finite volume $V$, is probed with the exponentially decaying incident field $\psi_{in}^\theta (u + sv) = e^{-Ks} e^{-Kv \cdot r}$, $K > 0$.

Figure 1. The data measurement configuration

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1 For simplicity and without loss of generality, we consider two-dimensional objects, that is objects which are independent of one coordinate axis
2 The index $\mu$ parameterizes the direction of the unit vector $v$ along which the incident field decays exponentially.
We make the assumption that the interaction of the incident field with the object results in the formation of a total field $\psi$ satisfying the differential equation

$$\nabla^2 \psi_\theta = -\phi \psi_\theta$$

(1)

where $\phi$ is the, so-called, object function that quantifies the object properties and structure. The solution to Eq.(1) can be formally expressed in integral form with use of Green function techniques [7] as

$$\psi_\theta(r) = \psi_{in}^I(r) + \psi_\theta^S(r)$$

(2)

where we have defined the scattered field $\psi_\theta^S$ via

$$\psi_\theta^S(r) = -\int d^2r' G(r-r')\phi(r')\psi_\theta(r')$$

(3)

and the appropriate Green function $G$ is given by [7]

$$G(u + sv) = \ln\frac{1}{s^2 + u^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{-p|u|}$$

(4)

We consider now an object function $\phi(r) = \phi_0(r - r_0)$, i.e., an object function that arises from a spatial shift by $r_0$ of the object function $\phi_0$. Indicating the field scattered by the object $\phi_0(r - r_0)$ with $\psi_\theta^S(r; r_0)$, we showed in [1] that

$$\psi_\theta(r; r_0) = e^{-K\nu r_0} \psi_\theta(r - r_0; 0)$$

(5)

where $\psi_\theta(r; 0)$ is the field scattered by the object $\phi_0(r)$. Eq.(5) defines a translation property that relates the fields scattered from a known object placed at different locations, when the object is probed with the same incident field.

We return now to the configuration in Fig. 1, where the object $\phi(r) = \phi_0(r - r_0)$ is probed with known fields $e^{-r\nu}, \nu > 0$ which decay exponentially in the direction of the $t$ axis and corresponding scattered field data are measured along the straight line $v \cdot r = s = s_0$, where $s_0$ is a fixed measurement offset from the coordinate axis origin. We assume that the data are measured for a number of probing directions $\theta$ in some finite set $\Theta$ and are modelled for each probing direction as

$$d(t, \theta) = c_\theta (t; r_0) + n_\theta(t) \equiv r_0 \ast \psi_\theta^S(t u + s v; r_0) + n_\theta(t)$$

(6)

where $c_\theta$ and $n_\theta$ are a convolutional filter and a stochastic process modelling filtering effects and noise addition that may be present in the measurement process.

Assuming the noise process a zero-mean Gaussian process which is white with respect to both the probing direction parameter $\theta$ and the measurement position $t$ and that object function $\phi_0$ completely known, our goal is to estimate the unknown location $r_0$ from the data $d(t, \theta), t \in R, \theta \in \Theta$. For that, we compute the corresponding log likelihood function, which under the given noise assumptions attains the form [8]

$$L(r_c) = \Re \sum_{\theta \in \Theta} \left\{ \int_{-\infty}^{\infty} dtd(t, \theta) \overline{c_\theta(t; r_c)} - \frac{1}{2} \int_{-\infty}^{\infty} dt \left| c(t; r_c) \right|^2 \right\}$$

(7)

where $r_c = t_c u + s_c v$ is a test location and $\Re$ indicates the real part. After computation of the log likelihood function, the estimate of the unknown location $r_0$ is taken to be the point $r_c$ of global maximum of the function, i.e. $R_c = \arg\max_r L$

Even though the second term in the log likelihood function in Eq.(7) is independent of the data and, therefore, can be precomputed and stored for a number of test locations $r_c$, the first term in the same equation can be computed with the use of efficient fast Fourier transform-based algorithms. Indeed, we show in [1] that where the overbar indicates the complex conjugate. Therefore, we see that the first term in the log likelihood

3 Unless explicitly denoted otherwise, all integrations are over the support volume $V$ of the object.
function in Eq.(7) can be computed via a tomographic procedure in which, for each probing direction, the measurements are convolutionally filtered and backpropagated with the backpropagation kernel $e^{i r_\ell \cdot \hat{e}}$ and then coherently summed. The convolutional filter is the filter matched to the scattered field produced by the given object $\phi_0$ when located at the origin of the coordinate system.

The maximum likelihood estimation procedure described in here is easily extended into maximum a posteriori estimation procedure by adding a third term to the right hand side of Eq.(7). This term is selected so as to incorporate any prior information that may be available for the estimation of the unknown object location and corresponds to a pre-computable data-independent term.

We have tested the robustness of our proposed maximum likelihood and maximum a posteriori location estimation procedures via computer simulation and report our findings in this paper. In our test, we evaluated the robustness of the two procedures to various levels of noise as well as the corresponding bias. The findings of our evaluation will be presented at the conference. Future relevant research may also follow the avenues of linear and nonlinear tomographic inversion of the partial differential equation $\nabla^2 \psi = -\phi \psi$ for estimation of the object function $\phi$ from measurements of the scattered fields $\psi_\theta$. These and other research avenues are currently being explored and the findings will be reported elsewhere.

REFERENCES