A HYBRID TECHNIQUE FOR THE ANALYSIS OF SCATTERING BY IMPEDANCE WEDGES

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Abstract: The paper presents an eigenfunction solution to the canonical problem of diffraction of an obliquely incident plane electromagnetic wave by a wedge with impedance boundary conditions. The expansion functions, which satisfy all the conditions of the boundary-value problem except for the radiation condition, are derived in an explicit form, whereas the expansion coefficients are numerically determined from the radiation condition. The diffraction coefficient is extracted from the series solution by employing the high-frequency asymptotic solution of the canonical diffraction problem.

INTRODUCTION
Diffraction coefficients of edges in non-metallic surfaces are important for accurate simulations of the electromagnetic scattering in a broad variety of microwave applications, ranging from the simulation of mobile communication channels and RCS of stealth vehicles to the antenna analysis. When the material properties of the scattering boundaries are modeled by the impedance boundary conditions (see, for example, Senior and Volakis [1]), the diffraction coefficient is deduced from the solution of a canonical problem of diffraction of a plane electromagnetic wave by an infinite impedance wedge.

In the case that the incidence direction and the wedge angle are arbitrary, an analytical solution of the problem does not seem possible. This paper shows that a quasi-analytical solution can be obtained by using a hybrid approach, called the method of edge functions (MEF) [2]. The method is based on expanding the solution in a series of edge functions (EFs), which, by their definition, satisfy the Maxwell equations, the edge, and the boundary conditions. The unknown expansion coefficients are found numerically from the condition that far from the edge, the solution consists of the ray-optical and edge-diffracted components.

THE BOUNDARY-VALUE PROBLEM
We assume that in the cylindrical coordinate system \((\rho, \phi, z)\), the faces of an infinite wedge are located at \(\phi = \pm \Phi\) with \(0 < \phi \leq \pi\), \(0 \leq \rho < \infty\), and \(-\infty < z < \infty\). Since the geometry is invariant under translation along the \(z\) axis, the two-component vector \(\tilde{U} = [E_\phi, Z_0 H_z] \mathbf{T}\), where \(Z_0\) is the free-space impedance, completely defines the electromagnetic field. The incident plane electromagnetic wave is given by

\[
\tilde{U}^{\text{inc}} = \tilde{U}_0 \exp \{ik_z \cos \beta - ik \rho \sin \beta \cos (\phi - \varphi_0)\}
\]

with \(\tilde{U}_0 = [E_\phi(z), Z_0 H_z(z)] \mathbf{T}\). The angles \(\varphi_0\) (\(|\varphi_0| < \Phi\)) and \(\beta\) (\(0 < \beta < \pi\)) define the incidence direction such that the latter lies in the plane bisecting the wedge when \(\varphi_0 = 0\) and in the plane perpendicular to the edge of the wedge when \(\beta = \pi / 2\). The \(z\) components of the total electric and magnetic fields, uniformly in \(\phi\) (\(|\varphi| \leq \Phi\)), satisfy the Helmholtz equation

\[
\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + k^2 \sin^2 \beta\right) \tilde{U} = \tilde{0} \quad \text{for} \quad 0 < \rho < \infty, \tag{2}
\]

the edge conditions

\[
|E_\phi|, |H_z| = O(1), \quad |\rho \nabla E_\phi|, |\rho \nabla H_z| \to 0 \quad \text{for} \quad \rho \to 0, \tag{3}
\]

and the radiation condition

\[
\frac{\partial}{\partial \rho} - ik \sin \beta \left(\tilde{U} - \tilde{U}^{\text{GO}}\right) \to 0 \quad \text{for} \quad \rho \to \infty. \tag{4}
\]

Here, \(\tilde{U}^{\text{GO}} = \left[ E_{\phi}^{\text{GO}}, Z_0 H_z^{\text{GO}}\right] \mathbf{T}\) is the geometrical-optics part of the field, which consists of the incident and reflected waves and is known in advance. The boundary conditions are the impedance boundary conditions:
\[
\frac{1}{\rho} \frac{\partial}{\partial \phi} \phi \left( \frac{\lambda}{\kappa} \right) \sin^2 \beta E_z = Z_0 \cos \beta \frac{\partial}{\partial \rho} H_z,
\]
\[
\frac{1}{\rho} \frac{\partial}{\partial \phi} \phi \left( \frac{\kappa}{\kappa} \right) \sin \beta Z_0 H_z = -\cos \beta \frac{\partial}{\partial \rho} E_z,
\]
with \( \kappa \) being the normalized face impedances at \( \phi = \pm \Phi \), respectively.

**THE EXPANSION FUNCTIONS**

To solve the boundary-value problem (1) – (5), we shall expand the total field \( \bar{U} \) in a series of two-component vector functions \( \{ \bar{\Psi}_e \}_{p=0}^\infty \) of the form:

\[
\bar{\Psi}_p = \sum_{q=0}^{\infty} J_{\nu q}(k \rho \sin \beta) \left[ a_q \exp \{i \phi (\nu p + q)\} + (-1)^p b_q \exp \{-i \phi (\nu p + q)\} \right] c_q \exp \{i \phi (\nu p + q)\} + (-1)^p d_q \exp \{-i \phi (\nu p + q)\},
\]

where \( \nu = \pi l (2\Phi) \) and \( J_{\nu q}(k \rho \sin \beta) \) are the Bessel functions. The representation (6) satisfies the Helmholtz equation (2) and the edge condition (3) regardless of the values of \( a_q, b_q, c_q, \) and \( d_q \). The boundary conditions (5) are satisfied if these parameters are solutions of the following recursion:

\[
A_q \tilde{X}_q + 2 \chi_q(q)B_{q-1} \tilde{X}_{q-1} + \chi_q(q)C_{q-2} \tilde{X}_{q-2} = 0, \quad q = 0, 1, \ldots
\]

Here, \( \tilde{X}_q = [a_q, b_q, c_q, d_q]^T \) and \( \chi_j(q) = 0 \) if \( q \leq j \) and \( \chi_j(q) = 1 \) if \( q > j \). The coefficients in (7) are:

\[
A_q = \begin{bmatrix}
    ie^{i\nu p} & -ie^{-i\nu p} & -\cos \beta e^{i\nu p} & -\cos \beta e^{-i\nu p} \\
    ie^{-i\nu p} & -ie^{i\nu p} & -\cos \beta e^{-i\nu p} & -\cos \beta e^{i\nu p} \\
    \cos \beta e^{i\nu p} & \cos \beta e^{-i\nu p} & ie^{i\nu p} & -ie^{-i\nu p} \\
    \cos \beta e^{-i\nu p} & \cos \beta e^{i\nu p} & ie^{-i\nu p} & -ie^{i\nu p}
\end{bmatrix},
\]

\[
B_q = \begin{bmatrix}
    -i \sin \beta e^{i\nu p}/\kappa & -i \sin \beta e^{-i\nu p}/\kappa & 0 & 0 \\
    i \sin \beta e^{-i\nu p}/\kappa & i \sin \beta e^{i\nu p}/\kappa & 0 & 0 \\
    0 & 0 & -i \sin \beta e^{i\nu p}/\kappa & -i \sin \beta e^{-i\nu p}/\kappa \\
    0 & 0 & i \sin \beta e^{-i\nu p}/\kappa & i \sin \beta e^{i\nu p}/\kappa
\end{bmatrix},
\]

whereas \( C_q \) results from (8) upon the substitution \( \cos \beta \rightarrow -\cos \beta \). The lowest-order equation in (7) with \( q = 0 \) is homogeneous and has two linearly independent solutions:

\[
\tilde{X}_0 = \begin{bmatrix}
    \sin^2 \beta \\
    (-1)^p (1 + \cos^2 \beta) \\
    0 \\
    -2i(-1)^p \cos \beta
\end{bmatrix}, \quad \tilde{X}_1 = \begin{bmatrix}
    0 \\
    2i(-1)^p \cos \beta \\
    \sin^2 \beta \\
    (-1)^p (1 + \cos^2 \beta)
\end{bmatrix},
\]

either of which can be used to generate a solution of the recurrence (7). We designate these two possible solutions as \( \tilde{\Psi}^e_q \) and \( \tilde{\Psi}^h_q \) and note that they describe the two basic polarization cases when \( \beta = \pi/2 \).

**DETERMINING THE EXPANSION COEFFICIENTS**

Having constructed the expansion functions, we can represent the total field as the series
\[ U(\rho, \varphi, z) = e^{j2\rho / \rho} \sum_{i=0}^{\infty} \left[ \lambda_{i}^{\perp} \tilde{\Psi}_{i}^{\perp}(\rho, \varphi) + \lambda_{i}^{h} \tilde{\Psi}_{i}^{h}(\rho, \varphi) \right] \]  \hspace{1cm} (11)

with the unknown expansion coefficients \( \lambda_{i}^{\perp, h} \) to be adjusted so as to make (11) comply with the radiation condition (4). To this end, we can approximately impose the radiation condition at a finite distance \( R \) from the edge, where \( R \) is sufficiently large to make the associated error negligible. The series (11) is then truncated at \( p = P_{\text{max}} - 1 \), where the truncation number \( P_{\text{max}} \) depends on the chosen value of \( R \) and the required accuracy. The truncated series includes \( 2P_{\text{max}} \) unknown coefficients that can be determined, for example, by enforcing the radiation condition at a set of equidistant points: \( \varphi = -\Phi + (j - 0.5)\Phi / P_{\text{max}} \) with \( j = 1, 2, \ldots, 2P_{\text{max}} \). Figure 1 compares the exact Maliuzhinets solution (solid lines) with an EF-based solution (dots) derived from the radiation condition imposed at \( kR = 4 \). The absolute error does not exceed 0.028 in this case.

**EXTRACTING THE DIFFRACTION COEFFICIENT**

To obtain the diffraction coefficient, one has to consider the solution (11) when \( k\rho \gg 1 \). The Bessel function series are known to be unsuitable for the far-field analysis because the number of terms to be accounted for grows proportionally to \( k\rho \). This complication can be overcome by exploiting high-frequency asymptotic solutions of the diffraction problem (1) – (5). Such solutions consist of the geometrical-optics and edge-diffracted components, and the diffraction coefficient is the amplitude of the edge-diffracted wave. By assuming that at a distance \( \rho = R \) the asymptotic approximation is sufficiently accurate, we can derive the diffraction coefficient from the given total and GO fields. Because of their better approximation quality, the uniform asymptotic representations have been found more preferable than the non-uniform ones. Figure 2 illustrates the accuracy of the procedure when the EF-based and uniform asymptotic solutions are matched at \( k\rho = 4 \).

**REFERENCES**
