A General Formulation for Analysing an Array of Arbitrarily Oriented Collocated Circular Loops

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Abstract: The use of entire domain sinusoidal basis functions to analyse circular loop antennas results in simple formulations for obtaining the modal currents and input admittance. This method has been previously applied to coaxial, parallel and orthogonal circular loop arrays. In this paper, we further extend the method to obtain a new general formulation which is valid for any arbitrarily oriented array of collocated circular loops. The formulation also does not place any restriction on the size of the individual loops and was tested for two cases involving non-identical loops. For both cases, the theoretical results of the analyses were in close agreement with the measured results.

Introduction
Circular loop antennas are fundamental radiating structures and extensive studies on their radiation and impedance characteristics have been reported [1]. These studies on single circular loop antennas have also been extended to circular loop arrays. The two main methods of analysing mutually coupled loop arrays have been the use of piecewise subsectional basis functions and entire domain basis functions. It is well know that the method of using piecewise subsectional basis functions is very general and can be used for almost all wire structures. This usefulness of piecewise basis functions has been well illustrated through their application to analysing superquadric loop arrays by Jensen et al. [2]. Purely circular loop antennas, however, pose a special case: they have a very simple and symmetrical geometry which unlike other structures lends them most naturally to the use of entire domain sinusoidal functions. The use of entire domain functions results in straightforward formulations without the need to discretize the geometry. They also allow a study of the antenna characteristics in terms of modal currents. Thus circular loop antennas provide a strong case for the use of entire domain sinusoidal functions and for these reasons have always been the main choice for investigating single loops as well as loop arrays. Consequently, in the past, simple formulations applicable for specific loop orientations have been provided through the use of entire domain functions to the cases of coaxial loops [3], parallel loops [4] and collocated orthogonal loops [5]. The aim of this paper is to further extend the use of sinusoidal entire domain basis functions to provide a new general formulation which is applicable to an array of arbitrarily oriented circular loops with a common centre or, in other words, are collocated.

General Formulation
The problem is formulated for a set of $N$ loops where the Eulerian angles $\alpha_{pq}$, $\beta_{pq}$, and $\gamma_{pq}$ [6] as shown in Figure 1 are used to represent the relative orientation of the Cartesian coordinate system of a loop $q$ with the coordinate system of a loop $p$. The loops are driven by slice voltage generators of amplitudes $V_p$ located on the loops at the azimuthal angles $\phi_{pq}$. The loops have a radii of $a_p$ and wire radii of $b_p$. Standard thin wire assumptions that the radii of the loops are large compared with the radii of the wires and the latter are small compared to the wavelength are made, that is: $b_p \ll a_p$ and $|kb_p| \ll 1$ where $k$ is the propagation constant.

Figure 1: Two Collocated Circular Loops

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On the surface of the loops, we require the total tangential electric field, \( E_{\phi p} \), to satisfy the boundary condition:

\[
E_{\phi p} = -\frac{V_p}{a_p} \delta(\phi_p - \phi_p^*) = \left[ \frac{1}{a_p} \frac{\partial \Psi_p}{\partial \phi_p} + jwA_p \right], \quad -\pi \leq \phi_p \leq \pi, \quad p = 1, 2, \ldots, N
\]

(1)

where \( \delta(\phi) \) is the Dirac's delta function, \( A_p \) is the magnetic vector potential resolved in the \( \phi_p \) direction in the coordinate system of loop \( p \) and \( \Psi_p \) is the scalar potential on loop \( p \). The currents in the loops are represented by a Fourier Series expansion:

\[
I_p(\phi_p) = \sum_{n=-\infty}^{\infty} I_n e^{-jn(\phi_p - \phi_p^*)}
\]

(2)

By expressing the potentials in (1) in terms of the loop currents and after a series of steps involving integration by parts and coordinate transformation, we arrive at the following set of linear equations:

\[
\begin{align*}
&f_{pq}^p = \frac{k a_p^2}{2} (s_{k+1} + s_{k-1}) - n^2 s_p^2 \\
g_{mn}^{pq} = \frac{k a_p^2}{4} (B_1^{pq} c_{m+1,n} + B_2^{pq} c_{m-1,n} + B_3^{pq} c_{m,n+1} + B_4^{pq} c_{m,n-1}) + \frac{mn}{k} e_{mn}^{pq}
\end{align*}
\]

(3)

where \( B_1^{pq} \) is given by the following column matrix:

\[
\begin{bmatrix}
-A_1^{pq} + jA_1^{pq} + jA_2^{pq} + A_2^{pq} \\
A_1^{pq} + jA_1^{pq} - jA_2^{pq} + A_2^{pq} \\
A_1^{pq} - jA_1^{pq} + jA_2^{pq} + A_2^{pq} \\
-A_1^{pq} - jA_1^{pq} + jA_2^{pq} + A_2^{pq}
\end{bmatrix}
\]

(4)

and \( A_{ij}^{pq} \) is the element at the \( i \)th row and \( j \)th column of the following coordinate transformation matrix:

\[
\begin{bmatrix}
\cos(\alpha_{pq} \cos(\gamma_{pq})) - \sin(\alpha_{pq} \cos(\beta_{pq} \sin(\gamma_{pq}))) - \cos(\alpha_{pq} \sin(\gamma_{pq})) - \sin(\alpha_{pq} \cos(\beta_{pq} \cos(\gamma_{pq}))) \\
-\sin(\alpha_{pq} \cos(\gamma_{pq})) + \cos(\alpha_{pq} \cos(\beta_{pq} \sin(\gamma_{pq}))) + \sin(\alpha_{pq} \sin(\gamma_{pq})) + \cos(\alpha_{pq} \cos(\beta_{pq} \cos(\gamma_{pq}))) \\
\sin(\beta_{pq} \cos(\gamma_{pq})) \\
\sin(\beta_{pq} \cos(\gamma_{pq}))
\end{bmatrix}
\]

(5)

(7)

where \( s_p^k \) and \( c_{mn}^{pq} \) are the Fourier expansion coefficients of the terms \( F_p \) and \( G_{pq} \) respectively, where:

\[
F_{pq}(\phi_p - \phi_p^*) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-jR_{pq}(\phi_p - \phi_p^*)} d\zeta
\]

(8)

\[
G_{pq}(\phi_p, \phi_q^*) = \frac{e^{-jR_{pq}(\phi_p, \phi_q^*)}}{R_{pq}(\phi_p, \phi_q^*)}
\]

(9)

\[
r(\phi_p - \phi_p^*, \zeta) = \left[ 4a_p^2 \sin^2 \left( \frac{\phi_p - \phi_p^*}{2} \right) + 4b_p^2 \sin^2 \left( \frac{\zeta}{2} \right) \right]^{0.5}
\]

(10)

\[
r_{pq} = \{(a_p + b_p)(A_1^{pq} \cos(\phi_q) + A_2^{pq} \sin(\phi_q)) - a_p \cos(\phi_q) \}^2
\]

(11)

\[
+ \{(a_p + b_p)(A_2^{pq} \cos(\phi_q) + A_1^{pq} \sin(\phi_q)) - a_p \sin(\phi_q) \}^2 + \{(a_p + b_p)(A_1^{pq} \cos(\phi_q) + A_2^{pq} \sin(\phi_q)) \}^2 \right)^{0.5}
\]

where \( \zeta \) describes the angular variation around the wire cross-section and \( A_{ij}^{pq} \) are as given in (7).

The modal current coefficients, \( I_n^p \), can now be easily obtained from a truncation of the linear equations in (3), followed by matrix inversion. Assuming a unit voltage excitation, the input admittance of each loop is then simply given by the summation of their modal current coefficients.
Results
The formulation was applied to two multiband scenarios involving loops $p$ and $q$ in one case and loops $p$, $q$ and $r$ in another case. The loop radii were chosen such that $k_d = 1.0$, $k_a = 0.8$ and $k_a = 1.2$. The wire radii were all equal and are given by $k_b = 0.0167$. The loop orientations in degrees were: $\alpha_{pq} = \alpha_{pr} = 0$, $\beta_{pq} =45$ and $\gamma_{pq} = \gamma_{pr} = 0$. The theoretical results were to be verified by measuring the self admittance of the half-loop equivalent of one of the loops. Consequently, loop $p$ was fed by a coaxial probe and the remaining loop/loops were shorted to ground. The measured self-admittance of loop $p$ was halved to compare with the computed theoretical results which were for an array of full-loops. The systematic errors in measurement were removed by comparing the measured and theoretical results of a single loop. For both the 2 loop and 3 loop cases, the theoretical and measured results (Figure 2) were seen to follow each other very closely.

![Figure 2: Measured and Theoretical Self Admittance of Loop p, a) Conductance b) Susceptance](image)

Conclusion
A new general formulation for analysing arbitrarily oriented collocated circular loop antennas has been developed. The antenna problem is completely described by expressing the relative orientation of the loops with respect to each other and the positions of the feed on each loop. The Fourier coefficients of the loop currents are easily obtained through matrix inversion. The method was used to analyse two sufficiently general cases involving 2 and 3 collocated loops of different sizes, representing multiband scenarios. The theoretical results were observed to correspond closely to the measured results, verifying the correctness of the analytical formulation.

References