HIGH-FREQUENCY WAVE DYNAMICS OF RAY-CHAOTIC SCATTERERS

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Abstract: Ray chaos, manifested by the exponential divergence of trajectories in an originally compact ray bundle, can occur even in linear electromagnetic propagation environments, due to the inherent nonlinearity of ray-tracing maps. In this paper, we present a compact summary of recent results on two-dimensional prototype test scenarios, with emphasis on possible implications for high-frequency wave dynamics ("ray-chaotic footprints").

INTRODUCTION AND SUMMARY OF PREVIOUS RESULTS

In electromagnetic (EM) propagation environments, ray chaos, i.e., exponential separation of nearby-originating ray trajectories, can be induced by certain geometrical features of reflecting boundaries (e.g., focusing/defocusing) as well as by constitutive ray-trapping properties of the medium, in view of the inherent nonlinearity of the ray-tracing maps. Remarkably, such complex behavior may be observed in relatively simple scenarios, the simplest paradigms being provided by “billiard-shaped” enclosures (see, e.g., Ott[1]). Although the onset of chaos is strictly forbidden in the (linear) wave (i.e., finite wavelength) regime, there is substantiated evidence that ray-chaotically-inclined systems, when observed at short (but finite) wavelengths, exhibit features which differ considerably from those associated with “regular” (e.g., coordinate-separability-induced) ray behavior. Ray chaos therefore seems capable of impressing clear, and often universal, “footprints” on the high frequency (HF) wave dynamics (see Berry[2] for a review of known results). Besides the inherent academic interest, ray chaos has been demonstrated to play a key role in a variety of engineering applications (see Galdi et al.[3] for a recent review). In Castaldi et al.[4], we explored a novel two-dimensional (2-D) ray-chaotic wave scattering configuration consisting of a planar dielectric layer with exponentially tapered refractive index profile backed by a smooth perfectly-electric-conducting (PEC) periodic undulating surface. The main novel features in this configuration are its internal/external nature, with simultaneous presence of both spatially-confined and exterior-outgoing rays and modes, and the key role of the permittivity profile in the onset of ray chaos (at variance with the pure conductor-vacuum dynamics in typical chaotic billiards). For this configuration, we performed a comprehensive ray analysis, which revealed the onset of typical chaotic behavior. We also carried out a rigorous full-wave analysis, which allowed for a comprehensive parametric study of the wave dynamics. Results for the HF regime indicated trends toward irregularity and other peculiar characteristics (not observed in geometries with “regular” ray behavior) which can be interpreted as “ray-chaotic footprints”. In the irregular (random-like, ergodic) regime, the wave dynamics was found to be effectively described by random-wave statistical models (see Castaldi et al.[4] for details). The performance characteristics produced by this model environment might be of interest in radar countermeasures and smart microwave absorbers.

CURRENT INVESTIGATION

Ray-Chaotic Cylindrical Configuration. In order to gain further insight into the HF scattering phenomenologies typical of ray-chaotic configurations, we are currently exploring a 2-D cylindrical version of the planar configuration in Castaldi et al.[4], whose finite extent in free space allows for characterization and assessment of (monostatic or bistatic) radar cross sections (RCS). In the \((r, \theta, z)\) cylindrical coordinate system in Fig. 1, this configuration is represented by a PEC \(z\)-independent smooth cylindrical boundary \(r = r_c(\theta)\) with periodic azimuthal corrugations of peak-to-peak height \(\Delta\), coated with a dielectric layer having a radially-decreasing (i.e., ray-trapping) graded-index profile

\[
n(r) = \sqrt{A - Br^2}, \quad A = \left(n_1^2 r_c^2 - r_c^2\right) \left(r_c^2 - r_1^2\right)^{-1}, \quad B = \left(n_1^2 - 1\right) \left(r_c^2 - r_1^2\right)^{-1},
\]
which is truncated at \( r = r_e + \Delta \) (with \( n(r_e) = 1 \)), and reaches its maximum value \( n_i \) at \( r = r_i \) on the interior boundary.

**Ray Tracing.** The ray equation in an unperturbed medium with refractive index profile (1) can be integrated analytically in closed form. Following Born and Wolf[5], it can be shown that the trajectory of a ray originating at \((r_0, \theta_0)\) with departure angle \( \alpha_0 \) (measured from the x-axis) is described by

\[
r(\theta) = \left[ \cos^2(\theta - \theta_0 + \phi) + \sin^2(\theta - \theta_0 + \phi) \right]^{1/2}, \quad R_{1,2} = \frac{u}{A \pm \sqrt{v^2 + w^2}}, \quad \phi = \frac{1}{2} \arctan \left( \frac{w}{v} \right),
\]

with \( u = \sqrt{2r_0n(r_0)}|\sin(\alpha_0 - \theta_0)|, v = 2n^2(r_0)\sin^2(\alpha_0 - \theta_0) - A, w = n^2(r_0)\sin[2(\alpha_0 - \theta_0)] \). Equation (2) defines a tilted ellipse centered at \((x=0, y=0)\) with semi-axes \( R_1 \) and \( R_2 \), thus yielding ray traces (in the dielectric layer region \( r_e(\theta) < r < r_i \)) with turning points at \( \theta = \theta_{opt,2} = \theta_0 - \phi \pm \pi/2 \), \( r = R_2 \). Ray interactions with the PEC boundary as well as those with the free-space interface at \( r = r_e \) need to be tracked separately by solving (either numerically or analytically) the arising nonlinear equations for the emerging rays. In our simulations, incident and escaping rays are parameterized by the angles \((\theta', \alpha')\) and \((\theta'', \alpha'')\) defining, respectively, the position of the impact point (modulo \( 2\pi \)) at the free-space interface and the incidence/departure directions (see Fig. 1). The multi-bounce ray dynamics inside the dielectric layer is parameterized via a phase-space-type diagram, displaying, for each encounter with the PEC boundary, the angular position \( \theta' \) of the impact point vs. the cosine of the corresponding departure angle \( \alpha'' \) with respect to the local tangent unit vector. Representative results from a comprehensive set of ray-tracing simulations are shown in Fig. 2, for a thin bundle of \( 10^3 \) incident rays with the parameter configuration listed in the figure caption. As one can see, the plots of the exit angle (Fig. 2(a)) and the corresponding departure direction (Fig. 2(b)) vs. the incidence angle display intermingled regions of regularity (corresponding to rapidly-escaping rays) and irregularity (corresponding to multibounce-trapped, exponentially diverging rays). As in Castaldi et al.[4], the exit angles and departure directions were found to be nearly uniformly distributed within the entire angular interval \([0, 2\pi]\), with the intermingled regular/irregular dependence on the incidence angle observed at arbitrarily small scales. Moreover, for a fixed value of \( n_i \), the “measure” of the regular regions was found to be almost independent of the value of the incidence angle, shrinking to zero as \( n_i \) was increased. The phase-space diagram in Fig. 2(c) clearly indicates that the multi-bounce ray trajectories evolving from the thin incident ray bundle tend to (asymptotically) cover the phase space uniformly (with possible exceptions of “regular islands” corresponding to periodic trajectories). The above results represent typical indicators of ray chaos, also observed in different ray-chaotic scattering configurations (e.g., the “3-disk pinball” in Kottos et al.[6]).

**Full-Wave Analysis.** As a first step toward revealing and characterizing possible “ray-chaotic footprints” in the HF wave dynamics of the configuration in Fig. 1, we are currently working on the development of a rigorous full-wave analysis for the scattering of a time-harmonic (\( \exp(j\omega t) \)) transverse-electric (TE) z-directed incident plane-wave field, \( E^i_z = E_0 \exp[-jk_0(x\cos\alpha' + y\sin\alpha')] \), with \( k_0 = \omega \sqrt{\mu_0 \mu_r} \) denoting the free-space wavenumber. For this cylindrical geometry, the incident field \( E^i_z \) and the reflected field \( E^r_z \) in the free-space region can be expanded in Fourier-Bessel series

\[
E^i_z(r, \theta) = E_0 \sum_{m=-\infty}^{\infty} J_m(k_0r) \exp[im(\theta - \alpha')], \quad E^r_z(r, \theta) = E_0 \sum_{m=-\infty}^{\infty} c_m H_m^{(2)}(k_0r) \exp(im\theta),
\]

with \( J_m \) and \( H_m^{(2)} \) denoting \( m \)-th order Bessel functions and Hankel functions of second kind, respectively, and with \( c_m \) denoting unknown coefficients. The field \( E^r_z \) transmitted into the layer can also be expanded in a Fourier series, synthesized in terms of wave functions which satisfy the Helmholtz equation inside the layer. This eventually leads to a countable infinity of radial differential equations which, for the profile in (1), can be solved analytically in closed form. One obtains
where \( \gamma = k_r \sqrt{B_r} \) and \( a_m, b_m \) are unknown coefficients. In (4), \( U(a,b,x) \) and \( L_n^m(x) \) denote the confluent hypergeometric function and the generalized Laguerre polynomial, respectively (Abramowitz and Stegun[7]). The unknown coefficients in (3) and (4) need to be computed by enforcing the relevant boundary conditions at the free-space interface \( r = r_e \) and at the PEC boundary \( r = r_i(\theta) \). We plan to use a class of point-matching techniques (Kleev and Manenkov[8]) for a computationally efficient and robust numerical implementation, so as to allow for a comprehensive parametric study of the RCS wave dynamics.

CONCLUSIONS

A compact summary of previous and new (preliminary) results on a special class of 2-D ray-chaotic EM wave scattering configurations has been presented here. In connection with the new cylindrical configuration, we note that the combined effects of periodic corrugations and dielectric fillings have already been explored (see, e.g., Kiang[9]) as a possible route for RCS reduction and control. For such applications, ray chaos may provide a different perspective.

REFERENCES