Network-Oriented Time Domain Green’s Functions for Periodic Arrays and Waveguides with Arbitrary Cross Sections

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1. Introduction

Interest in ultrawideband/short-pulse phenomena suggests that an analytic framework for any given scenario, when parameterized directly in the time domain (TD), might lead to better problem-matched physics, and thereby to better numerical convergence. For the class of periodic array and waveguide configurations in Fig. 1, these circumstances have motivated the present TD network-oriented prototype investigation, wherein (source-excited) $E_{(TM)}$ and $H_{(TE)}$-type modal Green’s functions (GFs) are treated individually, and propagate along TD transmission lines (TL), thereby accommodating possible planar vertically-stacked dielectric configurations within the same formalism. It is found that, in general, the TD field can be expressed in closed form in terms of causal TL GFs. However, for the waveguide with phase-shift walls and for phased periodic arrays (see Figs.1(a),(d)), the above $E$ and $H$ mode decompositions give rise to interesting causality issues: it is found that individually, each $E$ and $H$ mode is noncausal, and that it can be expressed in closed form in terms of a convolution between characteristic noncausal functions and the causal TL GFs (for the array case, the modes are Floquet wave (FW)-based). Causality on the total TD vector mode field is recovered by summing the $E$ and $H$ mode contributions. To learn the rules governing the vector TD modal behavior, we start in the frequency domain (FD), with subsequent inversion to the TD. For the infinite planar phased array, a numerical example of radiation due to short-pulse band-limited excitation demonstrates the accuracy of the TD modal-based algorithm, and also its efficiency, since only a few terms are required for describing the off-surface radiated field. Network-oriented results [1] for the nonphased array have already been used in a combined (TD-FW)-FDTD algorithm, shown in [2], for the analysis of periodic arrays of complex scatterers. Furthermore, the scalar TD GF [3] has already been used in practical fast TD method-of-moments algorithms for wide-band analysis of infinite periodic structures [4].

2. Statement of the Problem

The canonical problem for the network formulation under study includes the configurations in Fig. 1(a)-(c), excited by a horizontal electric dipole $J_i$ (bold face symbols denote vector quantities). The waveguide with phase-shift walls is equivalent to the infinite periodic array of electric dipoles in Fig. 1(d), with periodicities $d_x$ and $d_y$ along the $x$ and $y$ directions, respectively. The source point coordinates are denoted by $r = \rho + \gamma 1_x$, with $\rho = x 1_x + y 1_y$; for the observation point coordinates $r$, the primes should be omitted. Accordingly, individual sources in the $mn$-tagged periodic array are located at $r' + \rho_{mn}$, with $\rho_{mn} = md_x 1_x + nd_y 1_y$. Here, $1_x$, $1_y$, $1_z$ denote unit vectors along $x$, $y$, $z$, respectively. The modal FD and TD fields are related by the Fourier transform pair $f(r, \omega) = \int_{-\infty}^{\infty} f(r, t)e^{-j\omega t} dt$, $\tilde{f}(r, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(r, \omega)e^{j\omega t} d\omega$ in which $f$ can be either a scalar or a vector quantity; a caret ' denotes time-dependent quantities.

Boundary Conditions. Conducting walls imply that the electric field $\mathbf{E}$ tangent to the walls vanishes. In the FD, phase-shift walls for the rectangular waveguide imply that any field quantity at walls $x = d_x$ and $y = d_y$ is related to the field at the parallel pair of walls $x = 0$ and $y = 0$ via $\mathbf{E}(d_x, y, z) = \mathbf{E}(0, y, z)e^{-j\gamma_x d_x}$, and $\mathbf{E}(x, d_y, z) = \mathbf{E}(x, 0, z)e^{-j\gamma_y d_y}$, with $\gamma_x = \eta_x \omega/c$ and $\gamma_y = \eta_y \omega/c$ denoting phase-shift "wavenumbers"; here, $\eta_x$ and $\eta_y$ are normalized phasing quantities, $\omega$ is the radian frequency and $c$ is the ambient speed of light. As stated,

Fig. 1. Geometry of waveguides with different cross sections excited by an electric dipole (Figs.1(a)-(c)), and of a periodic array of sequentially excited dipoles (Fig.1(d)). The rectangular waveguide may have phase-shift lateral walls.
3. TD Green's Functions: Fourier Inversion of the FD Modal Expansion

The TD electric vector field \( E(r, r', \omega) \) is represented as \([5, p.444],[1] E(r, r', \omega) = \sum_n [E_n^E + E_n^H] \), where \( n \) is a generic problem-dependent double index, and the \( E \) and \( H \) nth modes are expressed as

\[
E_{t,n}^E(r, r', \omega) = k_{t,n}^E \left( k_{t,n}^E \cdot \mathbf{j}_1 \right) \Phi_n^E(\rho) \Phi_n^E(\rho') / \left( k_{t,n}^E \right)^2 Z_n^E(z, z', \omega)
\]

\[
H_{t,n}^E(r, r', \omega) = \left( k_{t,n}^H \times \mathbf{j}_1 \right) \left( k_{t,n}^H \times \mathbf{j}_1 \right) \Psi_n^E(\rho) \Psi_n^E(\rho') / \left( k_{t,n}^H \right)^2 Z_n^E(z, z', \omega)
\]

with \( \ast \) denoting complex conjugate. Here, \( Z_n^{E,H} \) are impedance-type TL-GFs as schematized in Fig.2(a), i.e., the voltage response at \( z \) excited by a unit current generator at \( z' \), propagating along \( z \) with the longitudinal propagation constant \( k_{t,n}^{E,H} \equiv \left( \omega^2/c^2 - \left( k_{t,n}^{E,H} \right)^2 \right)^{1/2} \) (see [5, p.193], [1] for details). The transverse vector wavenumbers \( k_{t,n}^{E,H} \), with \( k_{t,n}^{E,H} = \left( k_{t,n}^{E,H} - k_{t,n}^{E,H} \right)^{1/2} \), depend on the cross section geometry and boundary conditions. For waveguides with conducting walls, the transverse wavenumbers \( k_{t,n}^{E,H} \equiv \alpha_{t,n}^{E,H} \) are frequency-independent [5, pp.188-190], whereas for infinite arrays in free space or for phase-shift waveguides, the transverse wavenumbers are identical for \( E \) and \( H \) modes and possess also a frequency-dependent component, i.e., \( k_{t,n} = \alpha_{t,n} + \eta_{n} \omega/c \), where \( \eta_{n} \omega/c \) represents the phasing along \( L_{n} \). The scalar transverse modal functions \( \Phi_n(\rho) \) and \( \Psi_n(\rho) \) depend on the waveguide cross section and boundary conditions, and are specified in [5, p.191 and Sec.3.2] for a variety of geometries. For the array and phase-shift waveguide, instead of (1), we have

\[
\Phi_n(\rho) = \Phi_n(\rho)e^{-j\omega\eta_{n}L_{n}/c}/\sqrt{\eta_{n}} \Psi_n(\rho) = e^{-j\omega\alpha_{n}/c}/\sqrt{\eta_{n}} [5, p.251],[1].
\]

The total TD field, obtained from (1) via inverse Fourier transform from the FD, is correspondingly expressed as a superposition of TD modes \( E(r, r', t) = \sum_n [E_n^E + E_n^H] \). Implementation of the inversion requires prior identification of the singularities in the complex \( \omega \)-plane: branch points in \( k_{t,n}^{E,H}(\omega) \) and, for the phase-shift waveguide, poles in \( \left( k_{t,n}^{E,H} \right)^{-2} \). As shown in [1], a useful closed form expression for the TD electric field is

\[
E_{t,n}^E(r, r', t) = \overline{D}_{t,n}^E(t) \cdot \mathbf{j}_1 \Phi_n(\rho) \Phi_n(\rho') \odot Z_n^E(z, z', \tau)
\]

\[
H_{t,n}^E(r, r', t) = \overline{D}_{t,n}^H(t) \cdot \mathbf{j}_1 \Psi_n(\rho) \Psi_n(\rho') \odot Z_n^H(z, z', \tau)
\]

in which \( \odot \) denotes time convolution. For ordinary waveguides, \( \tau = t \), while for phased arrays or phase-shift waveguides, the retarded time \( \tau = t - \eta_{n}L_{n}/(\rho - \rho')/c \) appears in the TL GFs because of the \( \exp[-j\omega\eta_{n}L_{n}/(\rho - \rho')/c] \) factor in \( \Phi_n(\rho) \Phi_n(\rho') \). The TD dyads in (2) are obtained by Fourier-inverting a corresponding group of terms in the FD-FW expressions in (1). For example \( \overline{D}_{t,n}^E(t) = \frac{1}{i\omega} \int_{-\infty}^{\infty} [k_{t,n}^{E}(\omega) k_{t,n}^{E}(\omega)](k_{t,n}^{E})^{-2}(\omega)e^{i\omega} \, dw_{\omega} \), (recall
that for waveguides with conducting walls, $k_{z,n}$ is frequency-independent). The generic expression for the TD dyad is

$$\mathbf{D}_E(t) = \hat{a}_n(t)\mathbf{1}_u + \hat{b}_n(t)(\mathbf{1}_u\mathbf{1}_v + \mathbf{1}_v\mathbf{1}_u) + \hat{c}_n(t)\mathbf{1}_v\mathbf{1}_v$$

(3)

with TD coefficients $\hat{c}_n(t) = \frac{1}{2\pi i}(\alpha_n \cdot \mathbf{1}_u) e^{-j\beta_n \cdot t} = (\alpha_n \cdot \mathbf{1}_u) e^{-j\beta_n \cdot t}$ and $\hat{b}_n(t) = j(\mathbf{1}_u\mathbf{1}_v + \mathbf{1}_v\mathbf{1}_u)$.

For waveguides with conducting walls, where $\xi = 0$, the TD coefficients in the dyads reduce to $\hat{a}_n(t) = (\alpha_n \cdot \mathbf{1}_u)/\alpha_n^2$, $\hat{b}_n(t) = 0$, and $\hat{c}_n(t) = (\mathbf{1}_v\mathbf{1}_u)/\alpha_n^2$ (in this case, we can assume that $\mathbf{1}_u \equiv \mathbf{1}_v$ and $\mathbf{1}_v \equiv \mathbf{1}_u$), and the expressions in (2) become strictly causal. For waveguides with phase-shift walls, $\hat{c}_n(t)$ in (3) is a noncausal function since it also contributes for $t < 0$. The other dyad for the electric field in (2) is then evaluated by using the property $\mathbf{D}_E = -\mathbf{1}_u \times \mathbf{D}_E \times \mathbf{1}_v = \hat{c}_n(t)(\mathbf{1}_u\mathbf{1}_v - \mathbf{1}_v\mathbf{1}_u) + \hat{a}_n(t)\mathbf{1}_u\mathbf{1}_u$.

Though the TD-TL Green’s functions for waveguides are causal (see [1], [3]), the $H$-decomposed TD field expressions in (2) are noncausal for the phase-shift waveguide case, since they are determined by a time convolution between causal TD-TL Green’s functions and noncausal dyads that are time-spread around $t = 0$. However, as demonstrated in [1], causality is recovered by summing the modal $E$ and $H$ constituents. When the dipoles are subjected to short-pulse band-limited (BL) excitation $P(t)$, the field radiated by the array is given by the convolution of the waveform $P(t)$ with the impulsive fields. Each BL-TD $E$ and $H$ mode is given by the time convolution $\mathbf{E}_{BL}(r, r', t) = P(t) * \mathbf{E}_0(r, r')$.

4. Illustrative Example

Due to space limitations, we test the accuracy of the TD modal-based network formalism for a single example involving a pulse-excited planar phased array of dipoles. The simpler case of a waveguide with conducting walls ($\eta = 0$) avoids the time convolution in (2) and is thus numerically simpler. We have implemented the magnetic field solution, analogous to (2), with BL excitation. The result is compared with a reference solution obtained via element-by-element summation over the incident fields from all dipoles, i.e.,

$$\mathbf{H}(r, t) = \sum_{m,n=-\infty}^{\infty} \left[ \tilde{P}(t - t_{mn})/4(\pi R_{mn}^2/c) + \tilde{P}(t - t_{mn})/(4\pi R_{mn}^2) \right] (\mathbf{J}_x \times \mathbf{R}_{mn}),$$

where $R_{mn} = r - (r' + \mathbf{p}_{mn})$, $t_{mn} = \eta\mathbf{1}_u \cdot \mathbf{p}_{mn}/c + R_{mn}/c$, and $m,n = 0, \pm 1, \pm 2, \ldots$. The $mn$-series has been truncated when contributions from far-away elements are negligible, i.e., retaining elements with $|m|, |n| < 80$. The selected BL excitation is a normalized Rayleigh pulse $\tilde{P}(t) = \text{Re}[j/(j + \omega M/4)^2]$ (i.e., $\tilde{P}(0) = 1$), with FD spectrum $P(\omega) = 2i(\omega M)^{-1}(-j\omega^2/\omega M)\exp(-4\omega^2/\omega M)$ and central radial frequency $\omega M$, which corresponds to a central wavelength $\lambda_M = 2\pi c/\omega M$. We present here only the magnetic field results because the corresponding $T_{mn}^H(z,z',t)$ TL-GFs (see Fig.2(a)) are simpler than those for the electric field whose evaluation will be addressed in the future.

The dipoles of the array are oriented along $\mathbf{1}_y$, with periods $d_x = d_y = 0.1m$. The exciting waveform is chosen such that $\lambda_M = 2d_x$, so that the the average length of the pulse is twice the period of the array (see [3] for more details). Results for phasing $\eta = 0.7$ along the direction $\mathbf{1}_u \equiv \mathbf{1}_x$ are displayed in Fig.2(b). The magnetic field $\mathbf{H}_x$ is observed at the location $(x, y, z) = (0, 0, 10d_y)$, versus time $t$. It is remarkable that TD modes with wavenumbers $\alpha_n = 2\pi(n_1/d_x, n_2/d_x, n_3/d_y)$, $|n_1|, |n_2| \leq 2$, are adequate to represent the TD radiated field at any time $t$ in the plotted time interval.

The agreement with the element-by-element reference solution is excellent (as a further check on the numerics, both the TD modal expansion and the element-by-element reference solution yield a negligible $\mathbf{H}_x$ component). The total magnetic field is obtained by numerically summing its $E$ and $H$-mode constituents; it is seen that this sum cancels the small separate noncausal components (not shown), and renders the total signal causal.

5. Summary

TD modal expansions have been obtained for the configurations shown in Fig.1, using a unified formalism, and verified here by a preliminary single numerical example. A few TD modes are sufficient to reconstruct the field at any tested time and location. For phased arrays or waveguides with phase-shift walls, each $E$ and $H$ modal component is noncausal. Causality is recovered when summing the two nth modal $E$ and $H$ constituents.

References


