EVALUATION OF SINGULAR AND HYPERSINGULAR INTEGRALS FOR 2D GEOMETRIES USING A SINGULARITY CANCELLATION TECHNIQUE

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Abstract: A simple and efficient numerical procedure using a singularity cancellation method is presented for evaluating singular, near-singular, hypersingular and near-hypersingular integrals for 2D geometries. It is shown that the procedure not only has several advantages over the singularity subtraction method, but also improves on the singularity cancellation method utilizing the Duffy transformation. Analysis and results are presented for triangles and quadrilaterals.

INTRODUCTION

The integral equations of electromagnetics often involve singular potential integrals that require special numerical considerations for their evaluation. Unbounded (but integrable) singularities usually occur, for example, in the kernels of so-called self-terms in the method of moments whenever testing and source subdomains coincide. Quadrature rules for directly handling these singularities generally do not exist.

A singularity cancellation approach was presented [1,2] that is effective in computing both singular and nearly-singular integrals. In addition, the method was shown to have several advantages over the singularity subtraction and Duffy methods. In this presentation we summarize the previous results for singular and near-singular integrals and extend the method to hypersingular and near-hypersingular integrals. Analysis and numerical results will be presented for triangle subdomains.

SINGULAR AND NEAR-SINGULAR INTEGRALS

To evaluate potential integrals on triangular domains $D$, consider the projection of a nearby observation point $r$ onto a triangular element with vertices at $r_1, r_2, r_3$. From the projection point $r$, we subdivide the triangle into three subtriangles, as shown in Fig. 1, and evaluate the partial potential contribution from each. The analysis is detailed for subtriangle 1. We introduce a local $xyz$-coordinate system with origin at the projection point, corresponding to local vertex 1, as shown in Fig. 2. The potential integral over subtriangle 1 has the form

![Figure 1: Subdivision of a triangle into three subtriangles.](image1)

![Figure 2: Subtriangle coordinate system and geometry.](image2)
where $\mathbf{A}(\mathbf{r}')$ is a vector or scalar basis function, $R = \sqrt{(x')^2 + (y')^2 + z'^2}$ is the distance between source and observation points, and $e^{-\beta R}/(4\pi R)$ is the 3D Green’s function. Note that primes are used to indicate geometrical quantities on a subtriangle. To eliminate singularities or near-singularities associated with the $1/R$ term, we let

$$du = \frac{dx'}{R} = \frac{dx'}{\sqrt{(x')^2 + (y')^2 + z'^2}},$$

yielding $u(x') = \sinh^{-1}\left(x'/\sqrt{(y')^2 + z'^2}\right) = 1/2 \ln\left(R + x'/R - x'\right)$. We obtain (1) by summing contributions from all three subtriangles. The overall quadrature scheme is written as

$$\int_{S^1} \mathbf{A}(\mathbf{r}') \frac{e^{-\beta R}}{4\pi R} \, dS' = \int_{S^1} W_k \mathbf{A}(\mathbf{r}^{(k)}) \frac{e^{-\beta R}}{4\pi R} \, dS'$$

where $W_k$ are the weights for area coordinate sample points $\left(\mathbf{r}^{(k)}_1, \mathbf{r}^{(k)}_2, \mathbf{r}^{(k)}_3\right)$ of the original triangle of area $A$ and unit normal $\mathbf{n}$. In this form, the Green’s function may be replaced by any Green’s function with $1/R$ singularity; further, we essentially eliminate apparent differences between integration over smooth vs. non-smooth integrands (except that in the latter case, the weights and sample points depend on $\mathbf{r}_k$). In (3), $\mathbf{J} = 2A$ is the Jacobian of the transformation between global and parametric coordinates on the triangle and $W_k$ is a concatenated listing of dimensionless weight vectors

$$\left(\mathbf{\hat{n}} \cdot \mathbf{n}\right) \frac{W_k R \left(u^{(k)}_L - u^{(k)}_K\right) R^{(k)}}{\mathbf{J}}$$

arising from each subtriangle. The factor $\mathbf{\hat{n}} \cdot \mathbf{n}$ is included to extend the method to near-singularities that occur when the projected observation point falls outside the original triangle. In this case, at least one of the subtriangles lies entirely outside the source triangle. The factor $\mathbf{\hat{n}} \cdot \mathbf{n}$ makes its weights negative, and its contribution then cancels those from the partial domains of the subtriangles that extend outside the original triangle.

Numerical results are shown in Fig. 3 for a right triangle with vertices at $(0,0,0)$, $(0.1\lambda,0,0)$, $(0,0.1\lambda,0)$ and $\lambda = 10 \, [m]$. The figure shows the convergence behavior as the observation point moves a distance $d$ off of the triangle.

**Figure 3. Convergence results for near-singular points**

**HYPERSINGULAR AND NEAR-HYPERSINGULAR INTEGRALS**

To evaluate hypersingular potential integrals we subdivide a triangle into three subtriangles as shown in Figs. 1 and 2. From (1), the normal derivative of the potential integral over subtriangle 1 has the form
\[
\frac{\partial I}{\partial z} = \int_0^{\varepsilon} \int_{\phi_0(y)}^{\phi_1(y)} f(x', y', z) \frac{dy'}{R^3},
\]

where \( f(x', y', z) = -z \Lambda(r') (1 + jkr) e^{-jkr} / 4\pi \) is smooth and bounded. To eliminate singularities or near-singularities associated with the \( 1/R^3 \) term we let

\[
du = \frac{\rho^2 dx'}{R^3} \left(\frac{\rho^2 dx'}{((x')^2 + (y')^2 + z^2)}\right),
\]

where \( \rho = \sqrt{(y')^2 + z^2} \). This yields \( u = x' / \rho \) and (5) is now written as

\[
\frac{\partial I}{\partial z} = \int_0^{u} \int_{\phi_0(y)}^{\phi_1(y)} f(u, y', z) du dy',
\]

To eliminate the singularity associated with \( 1/\rho^2 \) we split and normalize the inner integral integral, and let\( \tan \theta_{LU} = x_{LU} / y' \) yielding

\[
\frac{\partial I}{\partial z} = \int_0^{\varepsilon} \int_{\phi_0(y)}^{\phi_1(y)} f(u, y', z) dy' - \int_0^{\varepsilon} \int_{\phi_0(y)}^{\phi_1(y)} f(u, y', z) dy',
\]

Where the subscripts \( L \) and \( U \) represent the lower upper limits, respectively. We then make the substitution

\[
dv_{LU} = \frac{y' \tan \theta_{LU}}{(y')^2 + z^2 \sqrt{\sec^2 \theta_{LU} (y')^2 + z^2}},
\]

and obtain

\[
v_{LU} (y') = \frac{1}{z} \tan^{-1} \left( \frac{(y')^2 + z^2 + z \tan \theta_{LU}}{1 + \tan^2 \theta_{LU}} \right),
\]

yielding a smoothed integrand. The near-singularities are handled using the method discussed in the previous section.

Figure 4 shows the convergence behavior as the observation point moves a distance \( d \) off of the triangle for the geometry given in the previous section.

![Figure 4. Convergence results for near-hypersingular points](image)

REFERENCES
