AN ELECTROMAGNETIC FIELD OF LINEAR MAGNETIC CURRENT IN AN IRREGULAR PLANE IMPEDANCE WAVEGUIDE

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Abstract: The exact Green function of the Helmholtz equation for a gradient impedance junction in a plane waveguide of fixed cross-section is derived with the help of the theory of the Riemann boundary-value problem. Green’s function is given in terms of expansion in two-parameter family of plane inhomogeneous waves. A number of possible applications are suggested, namely, the transformation ratios for eigenmodes of the regular part of the waveguide junction segment are derived analytically; the adiabatic approximation of the theory of smoothly irregular waveguides is estimated; the abnormally high transformation of the principal zero mode into the first mode is revealed even for an adiabatically smooth impedance junction.

INTRODUCTION

Among the problems solved in a primitive state of the Wiener-Hopf method, one problem was concerned with the propagation of acoustic waves in a plane waveguide with the stepwise-distributed surface impedance $Z(\sigma)$ of one of the waveguide walls (a junction of two semi-infinite regular impedance waveguides of like height) (Heins et al.[1]). In the work of Mittra et al.[2], exact solutions to electrodynamic versions of this problem have been obtained by the same method. As a source, in these works the eigenmode of a regular section of the waveguide has been used. In this paper, exact Green’s function of the Helmholtz equation for a rectilinear band $0 < z < d, -\infty < x < \infty$ with the inhomogeneous boundary condition of the third kind on one of its boundaries is constructed

$$
\left( \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + \omega^2 \varepsilon \mu \right) G' = -\delta(\hat{\rho} - \hat{\rho}_0),
$$

(1)

$$
\frac{\partial}{\partial z} G' = 0 \quad \text{for} \quad z = 0, \quad \frac{\partial}{\partial z} G' + i\omega Z(\sigma) G' = 0 \quad \text{for} \quad z = d.
$$

(2)

The coefficient $Z(\sigma)$ in this boundary condition (an impedance analogue for the permittivity of the familiar Epstein ‘transition’ layer, (Felsen et al.[3]))

$$
Z(\sigma) = \left( Z_1 + Z e^{-\sigma} \right) \left( Z + e^{\sigma} \right) (\sigma > 0), \quad Z = e^{i\sigma} \quad (-\pi < \varphi < \pi)
$$

(3)

is a complex-valued function describing the gradient transition from $Z(-\infty) = Z_1 = Z_1$ to $Z(+\infty) = Z_e = Z_e$. Its hodograph represents a circular arc having the angular size of $2\varphi$ and joining points $Z_1$ and $Z_e$. The solution obtained can be applied for the analysis of the electromagnetic field induced by a linear magnetic current in a gradient junction between two regular impedance waveguides. As the limiting case, this solution contains the stepped impedance distribution (Mittra et al.[2]).

METHOD OF SOLUTION

A boundary-value problem (1-2) is reduced to the integral equation of the second kind, for which, following the basic ideas of work Cherskii [4], an analytical solution is derived by reducing it to the Riemann problem of the linear conjugation of two analytical functions on the real axis with the help of the Fourier transform and conforming mapping. We seek the solution to problem (1–2) in the form of a sum

$$
G'(\hat{\rho}, \hat{\rho}_0) = G^a(\hat{\rho}, \hat{\rho}_0) + G(\hat{\rho}, \hat{\rho}_0),
$$

where

$$
G^a(\hat{\rho}, \hat{\rho}_0) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d(\eta, \tilde{z}, \tilde{z}_0)}{R(\eta)} e^{i(\varphi - \varphi_0)} d\eta
$$

(4)
is the solution to (1–2) with fixed $Z(\tau) = Z_\tau$ and $G(\tilde{\beta}, \tilde{\rho}_0)$ is the solution to homogeneous equation (1) with conditions (2). Here $R_\alpha(\eta) = v \sin \vartheta + iZ_\alpha \cos \vartheta$, $v = \sqrt{\kappa^2 - \eta^2}$, $Z_\alpha = Z_{\alpha \omega \epsilon \tau}$ ($\alpha = 1, 2$), $\kappa = k / \tau$, $d(\eta, \xi, \zeta_0) = \cos v \zeta \left[ \cos v (\delta - \xi_0) - iZ_\alpha \sin v (\delta - \xi_0) / \sqrt{\delta}, \right.$ \[\delta = d \tau, \xi_0 = \min(\zeta, \zeta_0), \zeta_\alpha = \max(\zeta, \zeta_0), \] $\bar{\chi} = \chi \tau$, $\bar{x} = \zeta \tau$, and $d$ is the waveguide height. The function $G(\tilde{\beta}, \tilde{\rho}_0)$ is represented as the double Fourier integral

$$G(\tilde{\beta}, \tilde{\rho}_0) = \frac{1}{4\pi} \lim_{\tau \to \infty} \int \frac{1}{X(\tau)} \lim_{\tau \to \infty} \int \frac{1}{R_\alpha(\tau)} \text{sh}(x_0 - x_1) \right),$$

where the integration surface $S$ coincides with the real plane $(x_1, x_2)$, exclusive of the straight line $x_1 = x_2$, which does not contribute the integral, the function

$$X(\eta) = \prod_{n=0}^\infty \gamma(\eta, \eta^\prime, \eta^\prime_0) = \Gamma(1 - i(\eta + \eta^\prime)) \Gamma(-i(\eta + \eta^\prime + \eta^\prime_0)), \quad \Gamma(\eta) \text{ stands for the gamma-function, and}$$

$$\eta^\alpha_0 = \sqrt{\kappa^2 - (\nu^\alpha)^2} \quad (\text{Im} \eta^\alpha_0 \geq 0),$$

where $\nu^\alpha_0$ are the roots of the following dispersion equation for a regular waveguide with the impedance $Z_\alpha$ of one of the waveguide walls:

$$\nu^\alpha_0 \nu^\alpha + d \bar{Z}_\alpha = 0 \quad (\alpha = l, r).$$

Let us transform integral representation (5), by employing the Cauchy-Poincaré theorem, into the residue serieses by deforming the integration surface $S$ in the space of two complex variables $C^2((z_1, z_2))$ into the Leray coboundary (Pham [5]) enclosing the analytical set of singularities of the integrand.

$$G(\tilde{\beta}, \tilde{\rho}_0) = \sum_{n, k=0}^\infty g^\alpha_{nk}(\tilde{\beta}, \tilde{\rho}_0) \cos \nu^\alpha_{nk} \bar{z} \exp i\eta^\alpha_{nk} \bar{z}$$

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where

$$g^\alpha_{nk}(\tilde{\rho}_0) = \pi(\tilde{\beta}, \tilde{\rho}_0) \psi_{nk} \sum_{p=0}^\infty (-1)^{p+1} \psi_{nk} \exp i\varphi(q-k) \frac{\exp \varphi(\eta^\alpha_{nk} + \eta^\alpha_{nk}')} {\text{sh}(\eta^\alpha_{nk} + \eta^\alpha_{nk})} \cos \nu^\alpha_{nk} \bar{z} \exp i\eta^\alpha_{nk} \bar{z},$$

$$g^\alpha_{nk}(\tilde{\beta}, \tilde{\rho}_0) = \pi(\tilde{\beta}, \tilde{\rho}_0) \psi_{nk} \sum_{p=0}^\infty (-1)^{p+1} \psi_{nk} \exp i\varphi(q+k) \frac{\exp \varphi(\eta^\alpha_{nk} + \eta^\alpha_{nk}')} {\text{sh}(\eta^\alpha_{nk} + \eta^\alpha_{nk})} \cos \nu^\alpha_{nk} \bar{z} \exp i\eta^\alpha_{nk} \bar{z},$$

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$$\psi_{nk} = [R, \eta) X(\eta) \eta = \eta_{nk}, \psi_{nk} = [R, \eta) X(\eta) \eta = \eta_{nk}.$$

The immediate substitution of (8), as well as of (5), into (1–3) evidences that we have found actually desired solutions.
The obtained Green function determines the electromagnetic field \( H_y = i \omega V G \), \( E_x = V \partial H_y / \partial z \), \( E_z = -V \partial H_y / \partial x \) excited by a linear magnetic current of density \( M = V e^{i \omega t} \delta(\vec{r} - \vec{r}_0) \) in a plane waveguide whose bottom wall is perfectly conducting while the surface impedance distribution of the upper wall is defined by (3).

If the source and the observation point are well off the irregular section of \( Z(x) \) distribution \(|x| > |\vec{x}| \gg 1\), then (8) becomes an expansion in terms of the systems of eigenmodes for regular waveguides:

\[
H_n^0(\vec{r}) = a_n^0 \cos \sqrt{n \pi} z \exp \left\{ \pm i \sqrt{n \pi} x \right\} \quad (\alpha = l, r; \; n = 0, \infty).
\]

Here the normalization \( a_n^0 = i \left[ R_n(\eta_n^0) \cos \sqrt{n \pi} \delta \right]^{1/2} \) has been chosen such that the energy transported by each mode does not depend on the indices \( n \) and \( \alpha \).

Taking into account that modes are orthogonal in these systems, we deduce that the transformation of the \( m \)-th mode of the left-side waveguide, in its irregular section, into the \( n \)-th modes of the right-side and left-side regular waveguides occurs with the transmission coefficient

\[
T_{mn} = \left\{ -\pi \frac{R_n(\eta_n^0) R'_m(\eta'_m^0)}{R'_n(\eta'_n^0) R'_m(\eta'_m^0)} \right\}^{1/2} X(\eta_n^0) \exp \left\{ \eta'_m - \eta'_n \right\} \frac{X(\eta'_m)}{\sinh(\eta'_m - \eta'_n)} \quad (n = 0, \infty) \tag{9}
\]

and the reflection coefficient

\[
R_{mn} = \left\{ \frac{\pi R_n(\eta_n^0) R'_m(\eta'_m^0)}{R'_n(\eta'_n^0) R'_m(\eta'_m^0)} \right\}^{1/2} X(\eta'_m) X(\eta_n^0) \frac{\exp(\eta'_m + \eta'_n)}{\sinh(\eta'_m + \eta'_n)} \quad (n = 0, \infty), \tag{10}
\]

where \((\alpha, \beta = l, r)\)

\[
R_\alpha(\eta_n^0) = i(\tilde{Z}_\alpha - \tilde{Z}_\beta) \cos \sqrt{n \pi} \delta (\alpha \neq \beta), \quad R_\alpha(\eta_n^0) = -\eta_n^0 \gamma_\alpha(\eta_n^0)^2 \cos \sqrt{n \pi} \delta, \quad \gamma_\alpha = \delta \left[ (\eta_n^0)^2 - \tilde{Z}_\alpha^2 \right] - iZ_\alpha.
\]

**CONCLUSION**

The exact Green function of the Helmholtz equation for a gradient impedance junction in a plane waveguide of fixed cross-section (an impedance analogue for the permittivity of the familiar Epstein ‘transition’ layer) is derived with the help of the theory of the Riemann boundary-value problem. A number of possible applications are suggested, namely, the transformation ratios for eigenmodes of the regular part of the waveguide junction segment are derived analytically; the adiabatic approximation of the theory of smoothly irregular waveguides is estimated; the abnormally high transformation of the principal zero mode into the first mode is revealed even for an adiabatically smooth impedance junction. Phenomena of last type are of great interest for clarifying the known effects of radio wave abnormal propagation in the Earth-ionosphere waveguide along the routes intersecting the terminator line (Walker [6]).

**REFERENCES**


