PERIODIC FRAME BASED DECOMPOSITION
OF FIELDS RADIATED FROM CYLINDRICAL SOURCES

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Abstract: A formulation is introduced for periodic windowed Fourier transform (WFT) frames. Such frames are then used to represent fields on cylindrical surfaces. Through phase matching at the surface, fields radiated from localized Gaussian windows are expressed in the form of general Gaussian beams, leading to a representation of radiated fields as sums of beam propagators. Such representations will be presented on test cases, and can be most useful to represent fields in complex environments, around linear sources or obstacles.

INTRODUCTION
Frame based decomposition, used so far for planar field distributions, provides a rigorous and flexible technique for Gaussian beam launching techniques. In some situations however, cylindrical sources or cylindrical equivalent surfaces may be preferred to planar ones. Around diffracting edges or linear antennas, for instance, a cylinder could be conveniently used as a matching surface to hybridize a rigorous numerical solution inside the cylinder, and Gaussian beam tracking outside the cylinder.

To perform Gaussian beam launching from a frame decomposition on a cylindrical surface, two problems must be solved:
- frame decomposition on cylindrical surfaces has to be formulated; fortunately, previous works on periodic wavelets have opened the way for a “periodic frame” formulation.
- the radiation of elementary Gaussian frame windows defined on a cylinder must be expressed in the form of beam propagators, which can be tracked locally in complex media and through multiple interactions.

PERIODIC FRAME FORMULATION
In order to express radiated fields as a summation of Gaussian beam fields, we seek to perform a decomposition of the initial distribution into a family of Gaussian windows. In the case of planar distributions, theory of frames ensures the stability of the decomposition, and a good localization of the significantly contributing windows [1]. To keep these appealing properties for the decomposition of a cylindrical distribution, we shall consider a periodic extension of a 1D distribution in the arc-length variable, with a period equal to the circumference of the cylinder. We shall apply a conventional WFT frame decomposition to the periodized function, and introduce the “periodic frame” on that basis.

In this section, we first outline the construction of a conventional Gabor frame in $L^2(\mathbb{R})$, before introducing “periodic frame” windows.

Conventional WFT frames. In the following, a WFT frame set will be denoted $\{\psi_{m,n}(x), m, n \in \mathbb{Z}\}$. $\psi_{m,n}$ is obtained by translating a single window function $\psi$:

$$\psi_{m,n}(x) = \psi(x - m\bar{x})e^{ink\bar{\xi}(x-m\bar{x})} \quad (1)$$

where $m$ is the index for spatial translations along $x$, with translation step $\bar{x}$, and $n$ is the index for spectral translations along the local spectral coordinate $k\bar{\xi}$ ($k$ is the wavenumber), with translation step $k\bar{\xi}$ in the spectral domain. In order for that family to be a frame, the following relation must be satisfied, if $\psi$ and its Fourier transform are reasonably localized: $\bar{x}k\bar{\xi} = 2\pi\nu$ with $\nu < 1$ [2].

A function $f(x) \in L^2(\mathbb{R})$ can then be expressed as:

$$f(x) = \sum_{m,n} a_{m,n}\psi_{m,n}(x), \quad \text{with} \quad a_{m,n} = \langle f, \varphi_{m,n}\rangle \quad (2)$$

where $\langle f, g \rangle = \int dx f(x)g^*(x)$, for $x \in \mathbb{R}$, defines the inner product in the function space $L^2(\mathbb{R})$. The set of functions $\{\varphi_{m,n}, m, n \in \mathbb{Z}^2\}$ is the dual frame. Its elements are deduced from a single dual function $\varphi$, through the same spatial and spectral translations as in (1).
Choosing a Gabor frame with Gaussian windows, \( \psi(x) \) can be taken as: \( \psi(x) = e^{-kx^2/2b} \), where \( b \) is a real positive parameter which defines the Gaussian decay of the windows. To optimize the computational efficiency of the frame decomposition, balanced frames are preferred, which implies: \( b = \bar{\varepsilon}/k\xi. \)

Periodic WFT frames. Let \( v \) be the arc-length variable along a smooth closed curve \( C \) with circumference \( C \). A scalar function \( f(v) \in L_2(0, C) \) is given along \( C \). We want to decompose the function \( f \) on a “periodic frame” \( \{\psi_{m,n}^P \in L_2(0, C), (m, n) \in [0, M-1] \times \mathbb{Z}\} \), where only a finite number, \( M \), of spatial translations is applied to the “periodic frame” initial window \( \psi^P \in L_2(0, C) \). Such a periodic frame decomposition would lead to the following expression:

\[
f(v) = \sum_{n \in \mathbb{Z}} \sum_{m=0}^{M-1} a_{m,n} \psi_{m,n}^P(v), \quad \text{with} \quad a_{m,n} = \langle f, \psi_{m,n}^P(0,C) \rangle.
\]

Here \( \langle \cdot, \cdot \rangle_{(0,C)} \) denotes the inner product in the function space \( L_2(0, C) \), and \( \{\psi_{m,n}^P \in L_2(0, C), (m, n) \in [0, M-1] \times \mathbb{Z}\} \) is the dual of the periodic frame \( \{\psi_{m,n}^P\} \).

This “periodic frame” decomposition is easily deduced from a conventional WFT frame decomposition of a periodic extension of the initial distribution \( f(v) \in L_2(0, C) \). Let us denote \( f^P(v), v \in \mathbb{R} \), the periodic extension of \( f \), with period \( C \), and \( a_{m,n}^P = \langle f^P, \varphi_{m,n} \rangle \) the frame coefficients of its decomposition on the conventional frame \( \{\psi_{m,n}\} \). Taking \( \bar{v} = C/M, M \in \mathbb{N} \), we have, for any \( l \in \mathbb{Z} \), \( a_{m+l,M,n}^P = a_{m,n}^P \).

The frame decomposition of \( f^P(v) \) leads then to an expansion (3) for \( f(v) \):

\[
f^P = \sum_{n \in \mathbb{Z}} \sum_{m=0}^{M-1} a_{m,n}^P \psi_{m,n}^P = \sum_{n \in \mathbb{Z}} \sum_{m=0}^{M-1} a_{m+1,M,n}^P \psi_{m+1,M,n}^P = \sum_{n \in \mathbb{Z}} \sum_{m=0}^{M-1} a_{m,n}^P \psi_{m,n}^P
\]

with \( a_{m,n}^P = a_{m,n}^P \) and \( \psi_{m,n}^P \) defined as:

\[
\psi_{m,n}^P(v) = \sum_{l \in \mathbb{Z}} \psi_{m+1,M,n}(v), \quad v \in [0, C].
\]

With \( \{\psi_{m,n}\} \) being a frame in \( L_2(\mathbb{R}) \), it can then be demonstrated that \( \{\psi_{m,n}^P\} \) is a frame in \( L_2(0, C) \), and that its dual frame \( \{\varphi_{m,n}^P\} \) is defined by:

\[
\varphi_{m,n}^P(v) = \sum_{l \in \mathbb{Z}} \varphi_{m+1,M,n}(v), \quad (m, n) \in [0, M-1] \times \mathbb{Z} \text{ and } v \in [0, C],
\]

where \( \{\varphi_{m,n}, (m, n) \in \mathbb{Z}^2\} \) is the conventional dual frame in \( L_2(\mathbb{R}) \). It follows that:

\[
a_{m,n}^P = \sum_{l \in \mathbb{Z}} \langle f^P(v + lC), \varphi_{m,n}(v + lC) \rangle_{(0,C)} = \langle f(v), \sum_{l \in \mathbb{Z}} \varphi_{m-lM,n}(v) \rangle_{(0,C)} = \langle f, \varphi_{m,n}^P \rangle_{(0,C)}
\]

\( \{a_{m,n}^P, (m, n) \in [0, M-1] \times \mathbb{Z}\} \) are thus the coefficients of a “periodic frame” decomposition of the distribution \( f(v) \) in \( L_2(0, C) \). The periodic frame is defined by (5), where \( \{\psi_{m,n}\} \) is a conventional WFT frame, with the additional constraint that its spatial translation step is an integer fraction of the circumference: \( \bar{v} = C/M, M \in \mathbb{N}. \)

RADIATION FROM FRAME DECOMPOSITIONS ON CYLINDERS

Let \( S \) be a cylindrical surface with axis \( Oz \), and circular cross section of radius \( a \). \( v = a\phi \) is the arc-length variable on \( S \), with \( (\rho, \phi, z) \) the usual cylindrical coordinates system. A scalar distribution \( f(v, z) \) corresponding to an outward propagating scalar (e.g. acoustic) field is given on \( S \).

A 2D frame for such a distribution is obtained by multiplying 1D frames for each of the variables. Along \( v \) the 1D frame is a periodic one and is denoted \( \{\psi_{m_1,n_1}^{\text{(I)}} \in [0, M-1] \times \mathbb{Z}\} \). The \( b \) parameter of the associated Gaussian window \( \psi^{\text{(I)}} \) is denoted \( b_1 \). Along \( z \), the 1D frame is a
conventional one and will be denoted \( \{ \psi_{m_2,n_2}^{(i)}(m_2,n_2) \in \mathbb{Z}^2 \} \), with \( b_z \) denoting the \( b \) parameter of the Gaussian window \( \psi^{(i)} \). The periodic 2D frame set is then: \( \{ \psi_{\mu}^{(i)}(v,z) = \psi_{m_1,n_1}^{(i)}(v) \psi_{m_2,n_2}^{(i)}(z), \) with \( \mu = (m_1,m_2,n_1,n_2) \in [0,M-1] \times \mathbb{Z}^3 \).

For a non periodic Gaussian window \( \psi_{\mu}(v,z) \), with \( v \in \mathbb{R} \), radiated fields can be described by the propagator \( B_{\mu}(\rho, \phi, z) \) whose expression is given by:

\[
B_{\mu}(\rho, \phi, z) = \frac{k^2 a}{(2\pi)^2} \int d\xi \int d\xi \, \psi_{\mu}(\xi, \xi) \exp[ik(\xi a \phi + \xi z)] \frac{H_{\mu}^{(1)}(k \rho)}{H_{\mu}^{(1)}(k a)}
\]

where \((\xi_1, \xi_2)\) are the local spectral variables associated to \((v, z)\) [3]. The tilde denotes the spectral distribution of \( \psi \), defined via \( \tilde{\psi}_{\mu}(\xi_1, \xi_2) = \int dv \, dz \psi_{\mu}(v, z) \exp[-ik(\xi_1 v + \xi_2 z)] \). \( H_{\mu}^{(1)}(k a) \) is the Hankel function of order \( ka \xi_1 \), and \( k_i = k \sqrt{1 - \xi_2^2} \).

In the case of the 2D periodic frame decomposition, the propagator for the periodic frame function \( \psi_{\mu} \) is:

\[
\begin{align*}
B_{\mu}^{(p)}(\rho, \phi, z) &= \sum_{l \in \mathbb{Z}} B_{m_1+lM,m_2,n_1,n_2}(\rho, \phi, z), \phi \in [0, 2\pi].
\end{align*}
\]

In practice, we assume that:

1— the support of the Gaussian window \( \psi^{(i)} \) and of its dual \( \varphi^{(i)} \) are much less than \( 2\pi a \), i.e. \( \sqrt{b_i/k} \ll 2\pi a \), so that \( B_{\mu}^{(p)} \sim B_{\mu} \) (mod. \( 2\pi \) in \( \phi \));

2— \( \psi_{\mu} \) is collimated: \( kb_i \gg 1 \) and \( kb_i \gg 1 \).

Under these assumptions, (8) can be evaluated asymptotically, yielding an expression of \( B_{\mu} \) in the form of a general Gaussian beam:

\[
B_{\mu}(r) = A \sqrt{\frac{\det \Gamma^{-1}(0)}{\det \Gamma^{-1}(s)}} \exp ik \left[ s + \frac{1}{2} x^T \Gamma(s) x \right], \quad \text{with} \quad \Gamma^{-1}(s) = \Gamma^{-1}(0) + s I
\]

and \( \Gamma(0) = \begin{pmatrix}
\left( \frac{1}{b_2} + \cos \theta \right) \cos^2 \alpha + \frac{i}{b_2} \sin^2 \alpha & \frac{1}{b_2} - \frac{i}{b_2} - \cos \theta \\
\frac{\cos \alpha \sin \alpha}{\cos \theta} & \frac{-\cos \alpha \sin \alpha}{\cos \theta} (\frac{1}{b_2} - \frac{i}{b_2} - \cos \theta)
\end{pmatrix}
\]

where \( \cos \theta = \sqrt{1 - ||\xi||^2} \) and \( \xi = ||\xi|| \frac{(\cos \alpha}{\sin \alpha}, (s, x_1, x_2) \) is a coordinate system associated with the beam, with origin \( O(m_1, \bar{v}, m_2 \bar{z}) \) on the cylindrical surface \( S \). \( s \) is the unit vector with projection \( \xi = (n_1 \hat{e}_1, n_2 \hat{e}_2) \) in the local tangent plane, oriented towards the direction of propagation, and \( \hat{x}_1 \) is in the “plane of incidence” \( (\hat{s}, \hat{p}) \), \( \hat{p} \) being the outward unit radial vector. In previous expressions, the overbar denotes the translation step associated with the corresponding spatial or spectral variable. \( I \) denotes the identity matrix.

The result in (11) conforms with local phase matching of the complex phases of the initial \( \psi_{\mu} \) distribution and of the emerging Gaussian beam (10) for \( ||\xi|| < 1 \). Consequently expression (10) with (11) for the beam propagators can be extended to general convex cylindrical surfaces in regions with sufficiently large local radii of curvature.

**CONCLUSION**

A frame decomposition of field distributions on cylindrical surfaces has been introduced, leading to the expression of radiated fields as a summation of Gaussian beam fields. Numerical illustration of the conditions of validity and accuracy of such field representations will be presented. Such representations can be used to represent source or diffracted fields within complex environments where beam propagators are especially useful to track fields locally.

**REFERENCES**

