Design of a Nonuniform Array for Joint Direction-of-Arrival and Range Estimation

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Abstract. This paper considers the design of a linear array of antennas used to jointly determine the range and direction-of-arrival (DOA) of a single emitter in the close proximity of the array. The Cramér-Rao bound (CRB) associated with this problem is derived and compared for some different configurations of the array elements. The results indicate that in order to minimize the CRB for the DOA, the sensors should be clustered at the ends of the array. On the other hand, to minimize the CRB on the range estimate, it is advantageous to spread out the sensors along the array baseline.

Introduction. We study an array that is used to determine both the range and the bearing to an emitter in its immediate vicinity. As we are interested in the fundamental, underlying estimation problem, we intentionally neglect effects resulting from mutual coupling between the antenna elements and from possible correlation of the sensor noises. We also assume that no multipath is present, and that the output signals from all sensors can be coherently processed.

Array model. The array under study comprises \( N \) omnidirectional sensors that receive the signals emitted by a single point source at range \( r \) and angle \( \theta \), as shown in Figure 1(a). If the source transmits the signal \( s(t) \), then the received \( N \)-vector at time \( t \) can be written

\[
y(t) = a(\theta, r)s(t) + e(t)
\]

where \( a(\theta, r) \) is the array response to a point source at range \( r \) and direction-of-arrival \( \theta \), and \( e(t) \) is a noise term. We assume that \( e(t) \sim N(0, \sigma^2 I) \), that \( e(t) \) is independent of \( e(t') \) for \( t \neq t' \), and that \( s(t) \) is a statistically white Gaussian random process with zero mean and unit power. The array response then takes on the following form:\footnote{Throughout the paper, \( (\cdot)^T \) is the transpose; \( (\cdot)^H \) stands for the conjugate transpose; \( I \) is the identity matrix; \( \Tr \{ \cdot \} \) denotes the trace of a matrix; and \( \mathbb{E} \{ \cdot \} \) is statistical expectation.}

\[
a(\theta, r) = \begin{bmatrix}
e^{j \frac{2\pi}{\lambda}(x_1-r)+j\psi} & \cdots & e^{j \frac{2\pi}{\lambda}(x_N-r)+j\psi}
\end{bmatrix}^T
\]

where \( \psi \) is the phase of the emitted signal. In (2), \( x_n \) is the distance from sensor \( n \) to the source, which can be found from the cosine theorem:

\[
x_n = \sqrt{r^2 + d_n^2 - 2d_n r \cos(\theta)}
\]

where \( \{d_n\} \) are the coordinates (with appropriate sign) of the sensors relative to the center point of the array. Note that we assume that the array is so close to the emitter that the commonly made planar-wave approximation is not valid.

Cramér-Rao bound (CRB). Suppose \( N_s \) independent time samples of \( y(t) \) are obtained. The CRB theorem then states that any unbiased estimates \( \hat{r}, \hat{\theta} \) of \( (r, \theta) \) must satisfy [1]

\[
\var(\hat{r}) \geq \text{CRB}(r) = [FIM^{-1}]_{1,1}, \quad \text{and} \quad \var(\hat{\theta}) \geq \text{CRB}(\theta) = [FIM^{-1}]_{2,2}
\]

where FIM is the Fisher information matrix, whose elements are defined as follows:

\[
[FIM]_{k,l} = N_s \Tr \left\{ \frac{\partial R}{\partial \theta_k} R^{-1} \frac{\partial R}{\partial \theta_l} R^{-1} \right\}, \quad k, l = 1, 2
\]
In (5), \( p_1 = r \) and \( p_2 = \theta \), and \( R \) is the covariance matrix of \( y(t) \):

\[
R = \mathbb{E} \{ y(t)y^H(t) \} = \alpha(\theta, r)\alpha^H(\theta, r) + \sigma^2 I.
\]

Under the assumptions made, we have that \( [R]_{k,l} = e^{j\frac{2\pi}{\lambda}(x_k-x_l)}e^{-j\frac{2\pi}{\lambda}(x_l-r)} + \sigma^2 \delta_{k,l} = e^{j2\frac{\pi}{\lambda}(x_k-x_l)} + \sigma^2 \delta_{k,l} \), and

\[
\frac{\partial[R]_{k,l}}{\partial r} = \frac{\partial}{\partial r} e^{j\frac{2\pi}{\lambda}(x_k-x_l)} = j2\frac{\pi}{\lambda} e^{j\frac{2\pi}{\lambda}(x_k-x_l)} \left( \frac{r - d_k \cos(\theta)}{\sqrt{r^2 + d_k^2 - 2d_k r \cos(\theta)}} - \frac{r - d_l \cos(\theta)}{\sqrt{r^2 + d_l^2 - 2d_l r \cos(\theta)}} \right). \tag{7}
\]

Similarly,

\[
\frac{\partial[R]_{k,l}}{\partial \theta} = j2\frac{\pi}{\lambda} e^{j\frac{2\pi}{\lambda}(x_k-x_l)} \left( \frac{d_k}{\sqrt{r^2 + d_k^2 - 2d_k r \cos(\theta)}} - \frac{d_l}{\sqrt{r^2 + d_l^2 - 2d_l r \cos(\theta)}} \right) r \sin(\theta) \tag{8}
\]

which completes the derivation of all ingredients in the CRB in (4)-(5).

**Array ambiguity function.** For given values of \( r \) and \( \theta \), say \( r_0 \) and \( \theta_0 \), the ambiguity function of the array is given by

\[
a(r, \theta) = |\alpha^H(r, \theta)\alpha(r_0, \theta_0)|. \tag{9}
\]

This function quantifies the sidelobes of the array. For the application that we consider, the signal-to-noise ratio (SNR) is so high that ambiguities do not really present a problem. Nevertheless, the ambiguity function is interesting to study because it provides a measure of how susceptible the array is to interference from angles very different from \( \theta \).

**Examples and discussion.** We considered an array with total aperture 1.5 m, and with \( N = 8 \) elements. The wavelength was \( \lambda = 0.16 \) m, corresponding to a 1.9 GHz carrier frequency. The SNR was 30 dB, and \( N_s = 100 \) samples were taken. Figure 1(b) shows seven different layouts of the array elements. For all layouts, the elements are located on a \( \lambda/4 \)-grid (except for layout 1, where the elements are located on a \( \lambda/2 \)-grid, and layout 3, where the elements are uniformly spaced).

In Figure 1(c), we show the \( \sqrt{\text{CRB}(r)} \) (range accuracy) and \( r \cdot \sqrt{\text{CRB}(\theta)} \) (cross-range accuracy) for the seven layouts, using \( r = 25 \) m and some different values of \( \theta \). In Figure 1(d), we show the corresponding result for \( r = 100 \) m. Figures 1(e)–(k) show the ambiguity functions for \( \theta \), with \( \theta_0 = 0 \) and \( r = r_0 = 100 \) m, associated with the different array layouts (the ambiguity function as a function of \( r \) is very smooth, and not shown here).

Our results indicate that to minimize CRB(\( r \)), one should spread out the sensors over the aperture. On the other hand, to minimize CRB(\( \theta \)), it appears favorable to place \( N/2 \) sensors at either end of the aperture and as closely spaced as possible.\(^2\) Indeed, there seems to be a tradeoff between the achievable accuracy for \( r \) and \( \theta \). In conclusion, the choice of array design depends on whether one prefers to maximize the accuracy of \( r \) or that of \( \theta \), and on other considerations such as the range of interest for \( \theta \) and the actual radiation pattern of the sensors. A more complete study of the array optimization problem, given a suitable criterion function, would be of interest.

**References**


\(^2\)This observation has been made before in [2], which considered the problem of estimating only the direction-of-arrival by using an antenna array. However, the problem of estimating \( \theta \) is quite different from that of jointly estimating \( (r, \theta) \).
Figure 1: Illustration of the possible performance of joint estimation of \( r, \theta \). Subfigures (e)–(k) show the ambiguity functions for the estimation of \( \theta \), for the different array layouts, evaluated at \( \theta_0 = 0^\circ \) and \( r = r_0 = 100 \text{ m} \).