PULSED BEAM SUMMATION FORMULATION FOR SHORT-PULSE RADIATION
BASED ON WINDOWED RADON TRANSFORM FRAMES

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Abstract: We introduce a novel formulation for short pulse radiation from an expanded aperture distribution, wherein the field is expanded, directly in the time-domain (TD) into a discrete lattice of shifted, tilted and delayed pulsed beams (PB). The PB excitation coefficients are extracted from the space-time data via a windowed Radon transform (WRT), that can be regarded as a windowed slant stack transform (W-SST). The formulation is based on a new class of frames, termed WRT frames. Using properly chosen isodiffracting PB window functions provides (a) the snuggest WRT frame; (b) analytically tractable PB propagators; and (c) simple window functions for processing the space-time data.

INTRODUCTION
The discrete phase-space beam summation formulation for ultra wideband (UWB) radiation from aperture sources presented in [1] has addressed the problem from a frequency domain (FD) point of view. The FD formulation has several attractive features that make it amenable for an extension into the TD: (a) the beam lattice is frequency independent; (b) the iso-diffracting Gaussian beam (ID-GB) basis provides the “snuggest” frame for all frequencies; (c) the parameters of the resulting GB propagators are frequency-independent. In the TD formulation presented here, the time-dependent field is expanded into a discrete lattice of shifted, tilted and delayed PB propagators (Fig. 1(a)). This spectrum of PBs is matched to the source data via the discrete windowed Radon transform (WRT), that can be regarded as a windowed slant stack transform (W-SST). The formulation is based on a new class of frames, termed WRT frames which are described below.

THE WRT FRAME FORMULATION
As in [1], the formulation is presented in the context of radiation into the uniform half space z > 0 due to a given UWB scalar field distribution \( u_0(x, t) \) in the \( z = 0 \) plane, where \( x = (x_1, x_2) \) are coordinates transverse to \( z \) in the 3D coordinate system \( r = (x, z) \). The radiated field is denoted as \( u(r, t) \). \( u_0(x, t) \) is essentially limited to the frequency band \( \Omega = (\omega_{\min}, \omega_{\max}) \), i.e., \( u_0(x, t) \in L_2^{\omega_{\min}} \), the Hilbert space of square summable distributions in \( (x, t) \in \mathbb{R}^2 \times \mathbb{R} \) with frequency spectrum limited within \( \Omega \). The formulation proceeds along the following stages.

(a) Phase space lattice: Our starting point is the 5D phase-space \( Y = (x, \xi, \tau) \), where \( \xi = (\xi_1, \xi_2) = (\sin \theta \cos \phi, \sin \theta \sin \phi) \) defines propagation direction of the wave constituents emitted by the aperture (see [1]) and \( \tau \) is the initiation time. We introduce the phase spaces lattice

\[
Y_{\mu, \rho} = (X_{\mu, \tau, \rho}) = (m \bar{x}, n \bar{\xi}, p \bar{\tau}) = (m_1 \bar{x}, m_2 \bar{x}, n_1 \bar{\xi}, n_2 \bar{\xi}, p \bar{\tau}), \quad \mu, \rho \in \mathbb{Z}^4 \times \mathbb{Z},
\]

(1)

where \( \mu = (m, n) = (m_1, m_2, n_1, n_2) \), \( X_{\mu} \) is the phase space lattice discussed in [1], whose unit cell dimensions \( (\bar{x}, \bar{\xi}) \) satisfy the completeness condition \( \bar{k} \bar{x} \xi = 2\pi \) at the reference frequency \( \bar{\omega} > \omega_{\max} \) which, as discussed in (6) below, is typically chosen as \( \bar{\omega} \geq 3\omega_{\max} \). This grid provides an overcomplete phase-space coverage for all \( \omega < \omega_{\max} \). \( \bar{\tau} \) will be defined in (3), below.

(b) WRT frame sets for functions in \( L_2^{\omega_{\min}}(\mathbb{R}^3) \): For a given space-time window function \( \psi(x, t) \), the WRT frame set \( \{ \psi_{\mu, \rho}(x, t) \} \) and its dual \( \{ \psi_{\mu, \rho}(x, t) \} \) are defined as

\[
\psi_{\mu, \rho}(x, t) = \psi(x - x_{\mu}, t - \tau - c^{-1} \xi_{\rho} \cdot (x - x_{\mu})), \quad \psi_{\mu, \rho}(x, t) = \bar{\tau} \varphi(x - x_{\mu}, t - \tau - c^{-1} \xi_{\rho} \cdot (x - x_{\mu})).
\]

(2)

The “dual window” \( \varphi(x, t) \) is constructed as follows: Starting with \( \tilde{\psi}(x, \omega) \), the FD counterpart of \( \psi(x, t) \) ( \( \tilde{\psi} \) denotes FD constituents), we calculate its exact WFT dual function \( \tilde{\psi}(x; \omega) \) for each \( \omega \in \Omega \). \( \psi(x, \omega) \) and \( \tilde{\psi}(x; \omega) \) give rise to two dual WFT frame sets for \( \omega \in \Omega \) as defined in [1]. \( \tilde{\psi}(x, \omega) \) is then extended arbitrarily outside \( \Omega \) and gradually tapered to zero there so that it gives rise to a smooth space-time window \( \varphi(x, t) \). Now, the temporal sampling interval should satisfies

\[
\bar{\tau} \leq 2\pi/(\omega_{\max} + \omega_{\rho_0}),
\]

(3)

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where $\omega_{\text{max}}$ is the highest frequency in $\psi(x, t)$ or $\varphi(x, t)$.

As sketched in Fig. 1(b), the frame elements in (2) are localized around the phase space lattice points $Y_{\mu, p}$, i.e., they are localized around the space-time points $(x_m, \tau_p)$ with spectral tilts $\xi_n$. This structure implies that they can be used as kernels in a windowed Radon transform in the $(x, t)$ domain, hence the term WRT frames (see below).

(c) WRT expansion: A given aperture source distribution $u_0(x, t)$ can be expanded as

$$u_0(x, t) = \sum_{\mu, p} a_{\mu, p} \psi_{\mu, p}(x, t), \quad a_{\mu, p} = \langle u_0(x, t), \varphi_{\mu, p}(x, t) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt \, u_0(x, t) \varphi_{\mu, p}(x, t) \tag{4}$$

Eq. (4) expresses the expansion coefficients $a_{\mu, p}$ as a projection of the space-time data $u_0(x, t)$ onto the window $\varphi_{\mu, p}(x, t)$. Referring to the discussion in the previous paragraph and to Fig. 1(b), this operator is identical to a windowed SST (or a WRT) of $u_0(x, t)$ with respect to the “analysis window” $\varphi$, sampled at the phase space lattice points $Y_{\mu, p}$.

Eq. (4) expresses $u_0(x, t)$ in terms of the “synthesis frame” elements $\psi_{\mu, p}(x, t)$ with excitation amplitudes $a_{\mu, p}$. This expression can be propagated to $z > 0$ simply by propagating $\psi_{\mu, p}(x, t)$, giving rise to “PB propagators” $B_{\mu, p}(r, t)$, all having essentially the same form, that emerge from the space-time point $(x_m, \tau_p)$ in the direction $\xi_n$ (Fig. 1(c)). The final expression for the field expresses the fields as a sum of all these contributions (Fig. 1(a))

$$u(r, t) = \sum_{\mu, p} a_{\mu, p} B_{\mu, p}(r, t). \tag{5}$$

For $|\xi_n| > 1$, $B_{\mu, p}$ decay away from the $z=0$ plane and may be neglected in the summation for $z > 0$.

ID-PB WINDOWS

A special class of analysis and synthesis windows functions are the ID-PBs, which are the TD counterparts of the ID-GB windows used in [1]. The ID-PBs are shown to be the optimal windows for the WRT frame theory presented here for the following reasons: (a) with a proper choice of the parameters they provide the snuggest frame representation for all $\omega \in \Omega$; (b) the corresponding phase-space propagators have closed form expressions that can readily be traced in the ambient environment.

The FD expression for the ID-GB window is $\hat{\psi}_{\text{ID}}(x; \omega) = \exp(-i|x|^2/2b)$ where the window is matched to the phase space lattice $X_{\mu}$ if $b = \bar{x}/\bar{\xi}$ [1; Eqs.(6)]. Choosing the reference frequency $\bar{\omega}$ to be $\gtrsim 3\omega_{\text{max}}$ as discussed after (1) allows the dual window to be approximated by $\hat{\phi}_{\text{ID}}(x; \omega) \approx (\omega/\bar{\omega})^3(2/\bar{x}^2)\hat{\psi}_{\text{ID}}(x; \omega)$ for all $\omega \in \Omega$ [1; Eq.(7)]. The TD windows are obtained by multiplying these expressions by the filters $\hat{f}(\omega)$ and $\hat{g}(\omega)$ to be discussed below, and inverting the results to the TD, giving

$$\psi_{\text{ID}}(x, t) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{i\omega t} \hat{f}(\omega) \hat{\psi}_{\text{ID}}(x; \omega) = \text{Re} \left\{ \hat{f}(t - i|x|^2/2bc) \right\}, \tag{6a}$$

$$\varphi_{\text{ID}}(x, t) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{i\omega t} \hat{g}(\omega) \hat{\varphi}_{\text{ID}}(x; \omega) \approx (2/\bar{\omega}^3\bar{x}^2) \text{Im} \left\{ \hat{g}''(t - i|x|^2/2bc) \right\}. \tag{6b}$$

where $\hat{f}(t)$ and $\hat{g}(t)$ are the analytic time-signals corresponding to $\hat{f}(\omega)$ and $\hat{g}(\omega)$, respectively, with $\hat{g}''(t)$ denoting third order derivative. Analytic signals are a useful tool in TD wave theory: see [2; Appx.A] for a summary of their properties and applications. They are defined by inverting to the TD the positive $\omega$ part of the spectrum, i.e.,

$$\hat{f}(t) \equiv \int_{0}^{\infty} d\omega \, e^{-i\omega t} \hat{f}(\omega), \quad \text{Im} \ t \leq 0. \tag{7}$$

Clearly, the integral converges for $\text{Im} \ t \leq 0$, thereby defining an analytic function there. The temporal windowing in (6) is thus due to the pulse shape of $\hat{f}(t)$ and $\hat{g}(t)$ while the $x$-windowing is due to the negative imaginary part of the time-argument in the analytic signals that increases quadratically with $|x|$, thus causing a decay of the signal. Finally, since the windows should constitute a dual set for all $\omega \in \Omega$, the filters $\hat{f}$ and $\hat{g}$ should satisfy $\hat{f}(\omega)\hat{g}''(\omega) = 1$ for $\omega \in \Omega$, whereas outside $\Omega$ they may be chosen quite arbitrarily. In view of the discussion after (2), they are chosen such that $\hat{f}(\omega)\hat{\psi}_{\text{ID}}(x; \omega)$ and $\hat{g}(\omega)\hat{\varphi}_{\text{ID}}(x; \omega)$ gradually taper to zero there in order to provide smooth TD windows.
Examples for the WRT frames with the ID-PB windows are shown in Fig. 2. They correspond to the UWB domain $\Omega$ with $\omega_{\text{max}} = 1$ and $\omega_{\text{min}} = 0$. The filters $\hat{g}$ and $\hat{f}$ are taken to be 1 inside $\Omega$. Outside $\Omega$ they taper linearly to zero in the bands $(\frac{1}{4}\omega_{\text{min}}, \omega_{\text{min}})$ and $(\omega_{\text{max}}, \frac{3}{4}\omega_{\text{max}})$. We also choose $\omega = 3$ and $b = 400$ (for collimated PB propagators, $b$ should satisfy $kb > 0$ for all $\omega \in \Omega$). Recalling that $k\bar{x}\xi = 2\pi$ and $b = \bar{x}/\xi$ we have $\bar{x} = \sqrt{2\pi b}/k = 28.9$. Further details concerning analytic properties and numerical implementation of the WRT-PB expansion scheme are given in [3].

CONCLUSIONS

We introduced a discrete Pulsed Beam (PB) summation scheme for short-pulse radiation from aperture sources. The theory is formulated directly in the time domain and is based on a new class of frames, the windowed Radon transform (WRT) frames. The expansion coefficients (the PB amplitudes) are calculated via a W-SST of the space-time data (see Eq. (4) and Fig. 1(b)). Explicit expressions were given in the case of the isodiffracting pulsed beam (ID-PB) windows, which provide the snuggest frame representation, and can be tracked analytically through the ambient environment.

REFERENCES