TRANSMISSION LINE REPRESENTATIONS FOR HIGHER ORDER MODES IN LOSSY WAVEGUIDES

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Abstract: In this contribution the representation of a set of \(N\) modes in a lossy waveguide consisting of one signal and one reference conductor by a coupled set of \(N\) transmission lines is investigated. Power conserving and reciprocity conserving transmission line representations are compared and it is shown that the latter can lead to a set of decoupled transmission lines.

INTRODUCTION

The circuit representation by a set of coupled transmission lines of the propagation in multiconductor waveguides has been the subject of extensive research [1]. At low frequencies the fields in such a waveguide have negligible longitudinal components, i.e. the fields are quasi-TEM. In that case the circuit representation follows unambiguously from Maxwell’s equations [2]. For higher frequencies things are less obvious and many possible solutions have been proposed in the literature. Most often these circuit representations assume that the same power is propagated in the waveguide and the transmission lines. It was shown that this could lead to a violation of reciprocity [3] and hence other representations based on the orthogonality of the eigenmodes were proposed. In the present contribution we investigate how these different transmission line representations behave when representing higher order modes in a waveguide consisting of one signal and one reference conductor.

THEORY

The time-harmonic fields in a waveguide can be expanded in the set of eigenmodes of the waveguide [1]. Let us write the fields of an eigenmode as \(e(r) = E(\rho)e^{-\gamma_{w}z}\) and \(h(r) = H(\rho)e^{-\gamma_{w}z}\) with \(\rho = xu_{x} + yu_{y}\) and \(\gamma_{w}\) the propagation coefficient of the eigenmode. For two eigenmodes with index \(i\) and \(j\) we define the integral

\[
R_{ij}^{w} = \frac{1}{2} \int_{S} [E_{i}(\rho) \times H_{j}(\rho)] \cdot u_{z} dS,
\]

where \(S\) is the cross-section of the waveguide. Orthogonality of the eigenmodes in a waveguide yields that \(R_{ij}^{w} = 0\) when \(i \neq j\). The complex cross-power \(P_{ij}^{w}\) between two eigenmodes is given by

\[
P_{ij}^{w} = \frac{1}{2} \int_{S} [E_{i}(\rho) \times H_{j}^{*}(\rho)] \cdot u_{z} dS.
\]

In general \(P_{ij}^{w} \neq 0\) when \(i \neq j\), except in a lossless waveguide.

A set of \(N\) coupled transmission lines is described in time-harmonic regime by

\[
\frac{d}{dz} v(z) = -\overline{Z} \cdot i(z), \quad \frac{d}{dz} i(z) = -\overline{Y} \cdot v(z),
\]

with \(\overline{Y} = j\omega C + G\) and \(\overline{Z} = j\omega L + R\) the circuit admittance and impedance matrices. The eigenmodes are \(v(z) = Ve^{-\gamma_{l}z}\) and \(i(z) = Ie^{-\gamma_{l}z}\) and they satisfy

\[
-\overline{Y} \cdot \overline{e} = -\overline{Z} \cdot \overline{I}, \quad -\overline{I} \cdot \overline{e} = -\overline{Y} \cdot \overline{V},
\]

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where \( \mathbf{V} \) and \( \mathbf{F} \) group the modal currents \( \mathbf{I}_i \) and voltages \( \mathbf{V}_i, i = 1, 2, ..., N \), columnwise and where \( \mathbf{Y} \) is a diagonal matrix with the corresponding propagation coefficients. The matrix \( \mathbf{R} = \mathbf{V} \cdot \mathbf{F} / 2 \) is diagonal due to orthogonality and the cross-power matrix \( \mathbf{J} = \mathbf{V} \cdot \mathbf{J} / 2 \) is only diagonal in lossless cases.

We will now derive expressions for the transmission line parameters \( \mathbf{Y} \) and \( \mathbf{Z} \) as a function of the modal fields in the waveguide. In order that waves in both the waveguide and the transmission line have the same velocity we demand that \( \mathbf{Y} = \mathbf{Z} \). In power based transmission line representations we also demand that \( \mathbf{J} = \mathbf{J} \). These two constraints give rise to the following expressions for \( \mathbf{Y} \) and \( \mathbf{Z} \):

\[
\mathbf{Y} = \frac{1}{2} \mathbf{F} \cdot \mathbf{Z} \cdot \mathbf{W} \cdot \mathbf{W}^{-1} \cdot \mathbf{F}^T, \quad \mathbf{Z} = 2 \mathbf{J}^T \cdot \mathbf{Z} \cdot \mathbf{W}^{-1},
\]

where \( \mathbf{J} \) still needs to be determined. In general the matrices \( \mathbf{Y} \) and \( \mathbf{Z} \) will not be symmetric meaning that the transmission line is non-reciprocal whereas the waveguide consists only of reciprocal materials. In order to avoid this discrepancy it was suggested in [3] to replace \( \mathbf{Y} \) by \( \mathbf{J} \) and \( \mathbf{Z} \) by \( \mathbf{J}^T \) resulting in reciprocity based transmission line representations with symmetric \( \mathbf{Y} \) and \( \mathbf{Z} \) given by

\[
\mathbf{Y} = \frac{1}{2} \mathbf{F} \cdot \mathbf{Z} \cdot \mathbf{W} \cdot \mathbf{W}^{-1} \cdot \mathbf{F}^T, \quad \mathbf{Z} = 2 \mathbf{J}^T \cdot \mathbf{Z} \cdot \mathbf{W}^{-1}.
\]

In a multiconductor waveguide with \( N \) signal conductors it is obvious how to define \( \mathbf{J} \) in current based transmission line models, see e.g. [3]. However, it is not clear how to define the elements of \( \mathbf{J} \) when there is only one signal conductor. We can still demand that the total current on the signal conductor in the waveguide is equal to the sum of all the currents on all the lines in the transmission line, i.e.

\[
\sum_{j=1}^{N} I_{ij} = \oint_c \mathbf{H}_i(\rho) \cdot d\mathbf{l},
\]

where \( c \) is the boundary of the signal conductor. This assumes that at the load and generator of the set of coupled transmission lines all the signal conductors are connected with each other. Obviously (7) defines only \( N \) conditions on \( \mathbf{J} \), leaving many degrees of freedom. One choice is to make \( \mathbf{J} \) a diagonal matrix such that

\[
I_{ii} = \oint_c \mathbf{H}_i(\rho) \cdot d\mathbf{l}.
\]

From (6) it then follows that \( \mathbf{Y} \) and \( \mathbf{Z} \) are diagonal matrices, i.e. that the set of transmission lines becomes a decoupled set of equations. Note that \( \mathbf{Y} \) and \( \mathbf{Z} \) following from (5) will not be diagonal.

**NUMERICAL EXAMPLE**

As a numerical example we consider a coaxial cable with a lossy core conductor. The coaxial structure allows a semi-analytical treatment [4]. Consider the geometry of Fig. 1 consisting of a core cylinder with radius \( a = 0.0059 \) m and parameters \( \epsilon_2 = -j\nu\epsilon_0, \mu_2 = \mu_0 \) surrounded by another cylindrical medium with parameters \( \epsilon_1 = 2.1\epsilon_0, \mu_1 = \mu_0 \) and radius \( b = 0.02 \) m. The whole structure is covered by a perfect electric conducting mantle. In the limit for \( \nu \to +\infty \) a lossless coaxial cable is obtained. We chose a frequency of 10 GHz and we consider the fundamental mode and one higher order \( \phi \)-independent TM-mode. When the core becomes perfectly conducting the fundamental mode becomes a TEM mode. Fig. 2a and Fig. 2b show the elements of the \( \mathbf{Y} \) and \( \mathbf{Z} \) matrices in the PI-model (eqs. (5)) and the RI-model (eqs. (6)) as a function of \( \nu \). The non-reciprocity of the PI-model is clearly visible. When \( \nu \) becomes large \( L_{11} \) in both models reaches its value of a coaxial cable given by \( L = \mu \log(b/a)/(2\pi) = 244.2 \) nH/m. Note also that the elements of \( \mathbf{Y} \) and the off-diagonal elements of \( \mathbf{Z} \) decrease when \( \nu \) increases. For large values of \( \nu \) the difference between the RI-model and the PI-model vanishes.
CONCLUSION

Power and reciprocity based transmission line representations of higher order modes in waveguides were compared.

![Coaxial cable with lossy core.](image)

Figure 1: Coaxial cable with lossy core.

![Power and reciprocity based transmission line representations of higher order modes in waveguides.](image)

Figure 2: $\mathcal{L}$ (a) and $\mathcal{R}$ (b) for the PI-model (solid lines) and the RI-model (dashed lines) ($\circ$: $L_{11}$, $R_{11}$, $\Box$: $L_{22}$, $R_{22}$, $\times$: $L_{21}$, $R_{21}$, $+$: $L_{12}$, $R_{12}$).

REFERENCES

[4] F. Olyslager, A. Franchois and D. De Zutter, “Reciprocity based transmission line equations for higher order modes in lossy waveguides,” Submitted to *Wave Motion*. 