EXACT DESCRIPTION OF ELECTROMAGNETIC WAVES IN TERMS OF RAYS

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Abstract: New representations are proposed for vector electromagnetic waves in free space (or homogeneous media) of any state of coherence. These representations take the form of a radiance, i.e. the weight of a ray. The intensity, energy density or Poynting vector at any point correspond simply to the sum of the weights of all the rays through that point.

INTRODUCTION
The objective of this work is to show that a general traveling monochromatic or partially coherent electromagnetic vector field in free space (or homogeneous medium) can be represented exactly in terms of rays. In this representation, a weight is assigned to each ray, such that the integral of these weights for all the rays that go through a point gives exactly a physical field property, like the electric, magnetic, or total energy density, or the Poynting vector. The conservation of the weights along each ray is exact, regardless of wavelength or state of coherence; the only condition is the absence of evanescent wave components.

ELECTROMAGNETIC WAVES
For a free monochromatic field, the electric and magnetic vectors can be written as

$$E(r,t) = U(r) \exp(-i\omega t), \quad B(r,t) = V(r) \exp(-i\omega t),$$

where $\omega$ is the angular frequency. Maxwell’s equations are then:

$$\nabla \cdot U(r) = 0, \quad \nabla \cdot V(r) = 0, \quad \nabla \times U(r) = i\omega V(r), \quad \nabla \times V(r) = -i\omega c^{-2} U(r).$$ (1)

The electric and magnetic energy densities and the Pointing vector for these fields are defined as

$$E_{\text{u}}(r) = \frac{c}{8\pi} |U(r)|^2, \quad E_{\text{M}}(r) = \frac{1}{8\pi \mu_0} |V(r)|^2, \quad S(r) = \frac{1}{4\pi \mu_0} \text{Re}[U^*(r) \times V(r)].$$ (2a,b,c)

When there are no evanescent components, general solutions to Maxwell’s equations (1) for the electric and magnetic fields are superpositions of linearly-polarized plane waves of the form

$$U(r) = \sum_{p=\pm 1} \int_{4\pi} q(u,p) w(u,p + \frac{\pi}{4}) \exp(i k u \cdot r) d\Omega, \quad V(r) = \frac{1}{c} \sum_{p=\pm 1} \int_{4\pi} q(u,p) w(u,p + \frac{\pi}{4} + \frac{\pi}{2}) \exp(i k u \cdot r) d\Omega,$$ (3a,b)

where $q(u,p)$ is the complex amplitude or angular spectrum of the linearly polarized plane wave traveling in the direction of the unit vector $u$ and whose electric and magnetic vectors are in the directions $w(u,p + \frac{\pi}{4})$ and $u \times w(u,\theta) = w(u,\theta + \frac{\pi}{2})$, respectively. Here, the unit vector $w(u,\theta)$ is defined to be perpendicular to $u$ and at an angle $\theta$ from some reference direction, as shown in Fig. 1.

RAYS AND THE RADIANCE
The ray description of an optical field is comparatively very simple. The main quantity that describes the field is the radiance $B(r,u)$, which is the weight of a ray that goes through a point $r$ and travels in the direction of the unit vector $u$. The condition satisfied by the radiance is its conservation along rays. That is:

$$u \cdot \nabla B(r,u) = 0.$$ (4)

The radiometric analogues of the quantities in Eqs. (2) are the intensity and flux density given by

$$I(r) = \int B(r,u) d\Omega, \quad F(r) = c \int u B(r,u) d\Omega.$$ (5a,b)

The factor of $c$ is included in Eq. (5b) for dimensional reasons.

The simplicity of this model is usually regarded as a consequence of considering one or several limiting conditions and approximations, like small wavelength, small coherence length, quasi-homogeneous fields, scalar fields, paraxial propagation, etc. Our goal is to express the electromagnetic field exactly in terms of rays, i.e. as a radiance which is exactly conserved along rays, and that presents projection properties like the ones presented in Eqs. (5) regardless of wavelength, coherence properties, or numerical aperture.
ELECTRIC AND MAGNETIC RADIANCE ANALOGUES

Let us start by trying to write the electric energy density (i.e. the optical intensity) in a form similar to Eq. (5a). By substituting Eq. (3a) into Eq. (2a), we get

\[ E_E(r) = \frac{E_0}{8\pi} \sum_{p_1, p_2} \int \int \left(\psi^*(\mathbf{u}_1, p_1) \psi(\mathbf{u}_2, p_2) \mathbf{w}(\mathbf{u}_1, p_1, \frac{\alpha}{2}) \cdot \mathbf{w}(\mathbf{u}_2, p_2, \frac{\alpha}{2}) \exp[i(k(\mathbf{u}_2 - \mathbf{u}_1) \cdot \mathbf{r})] d\Omega_1 d\Omega_2 \right) \]  

(6)

\[ \mathbf{u}_1 = \mathbf{u} \cos \frac{\alpha}{2} + \mathbf{w}(\mathbf{u}, \theta) \sin \frac{\alpha}{2}, \]  

(7)

where \( \mathbf{u} \) is integrated over all directions, \( 0 \leq \alpha \leq \pi \), \( 0 \leq \theta < 2\pi \), and \( d\Omega_1 d\Omega_2 = \sin \alpha d\alpha d\Omega \). By doing this and inverting the order of integration and summation, Eq. (6) can be written in a form identical to Eq. (5a):

\[ E_E(r) = \int B_E(r, \mathbf{u}) d\Omega, \]  

(8)

where the electric radiance analogue \( B_E \) is defined as

\[ B_E(r, \mathbf{u}) = \frac{E_0}{8\pi} \sum_{p_1, p_2} \int \int \psi^* \left[ \mathbf{u} \cos \frac{\alpha}{2} - \mathbf{w}(\mathbf{u}, \theta) \sin \frac{\alpha}{2}, p_1 \right] \psi \left[ \mathbf{u} \cos \frac{\alpha}{2} + \mathbf{w}(\mathbf{u}, \theta) \sin \frac{\alpha}{2}, p_2 \right] \]

\[ \times \mathbf{w} \left[ \mathbf{u} \cos \frac{\alpha}{2} - \mathbf{w}(\mathbf{u}, \theta) \sin \frac{\alpha}{2}, p_1, \frac{1}{2} \right] \cdot \mathbf{w} \left[ \mathbf{u} \cos \frac{\alpha}{2} + \mathbf{w}(\mathbf{u}, \theta) \sin \frac{\alpha}{2}, p_2, \frac{1}{2} \right] \]

\[ \times \exp \left[ 2i k \mathbf{r} \cdot \mathbf{w}(\mathbf{u}, \theta) \sin \frac{\alpha}{2} \right] \sin \alpha d\theta d\alpha. \]  

(9)

It is trivial to see that, since in this definition \( \mathbf{r} \) appears exclusively in an inner product with a vector perpendicular to \( \mathbf{u} \), the transport equation in Eq. (4) is satisfied exactly:

\[ \mathbf{u} \cdot \nabla B_E(r, \mathbf{u}) = 0. \]  

(10)

This function therefore satisfies the main properties of the radiance, for any wavelength. An analogous definition \( B_M \), which is also exactly conserved along rays, can be found for the magnetic energy density, where the only difference is the replacement of \( p_1 \frac{1}{2} \) by \( p_1 \frac{1}{2} + \frac{\gamma}{2} \) in the arguments of \( \mathbf{w} \). A definition corresponding to the total energy density is given by the sum of \( B_E \) and \( B_M \).
RADIANCE ANALOGUE FOR THE POYNTING VECTOR

Finally, we look for a radiance analogue coupled to the Poynting vector by a relation of the form in Eq. (5b). By substituting Eqs. (3) into Eq. (2c), the pointing vector can be rewritten as

\[ \mathbf{S}(r) = \mathbf{S}_R(r) + \mathbf{S}_V(r), \]  

(11)

where

\[ \mathbf{S}_R(r) = \sum_{p_i, p_j, n = 1}^{2} \int_{4\pi} \psi^*(\mathbf{u}_1, p_i) \psi(\mathbf{u}_2, p_j) \frac{8\pi \mu_0 c}{d\Omega_1 d\Omega_2} \exp[ik(\mathbf{u}_2 - \mathbf{u}_1) \cdot \mathbf{r}][w(\mathbf{u}_1, p_i \frac{\pi}{4}) \cdot w(\mathbf{u}_2, p_j \frac{\pi}{4})] d\Omega_1 d\Omega_2, \]  

(12a)

\[ \mathbf{S}_V(r) = -\frac{1}{8\pi \mu_0 c} \sum_{p_i, p_j, n = 1}^{2} \int_{4\pi} \psi^*(\mathbf{u}_1, p_i) \psi(\mathbf{u}_2, p_j) \exp[ik(\mathbf{u}_2 - \mathbf{u}_1) \cdot \mathbf{r}] \times [w(\mathbf{u}_1, p_i \frac{\pi}{4})[w(\mathbf{u}_2, p_j \frac{\pi}{4}) \cdot w(\mathbf{u}_1, p_i \frac{\pi}{4})] d\Omega_1 d\Omega_2 - \mathbf{U}(r) \times \mathbf{B}(r) \]  

(12b)

Again, by using the change of variables in Eq. (7), we find that \( \mathbf{S}_R \) can be written in the form of Eq. (5b), in terms of a radiance analogue that is exactly constant along rays:

\[ \mathbf{S}_R(r) = c \int_{4\pi} \mathbf{B}_k(r, \mathbf{u}) d\Omega, \quad \mathbf{u} \cdot \nabla \mathbf{B}_k(r, \mathbf{u}) = 0. \]  

(13)

It turns out, however, that the corresponding expressions for \( \mathbf{S}_V \) are in terms of a vector radiance:

\[ \mathbf{S}_V(r) = \int_{4\pi} \mathbf{B}_V(r, \mathbf{u}) d\Omega, \quad (\mathbf{u} \cdot \nabla) \mathbf{B}_V(r, \mathbf{u}) = 0. \]  

(14)

The interpretation of the Poynting vector as a local measure of electromagnetic energy flux comes from the continuity equation, which only involves the divergence of this vector. One therefore is free to add or substract any curl to this vector without changing its interpretation. Because \( \mathbf{S}_V \) given in Eq. (12b) is exactly a curl, \( \mathbf{S}_R \) is as meaningful a measure of flux as \( \mathbf{S} \), so \( \mathbf{S}_V \) and therefore \( \mathbf{B}_V \) can be ignored.

CONCLUSIONS

The representations given here are the analogues for vector fields of generalized radiances for scalar fields proposed earlier [1,2]. While the treatment above assumes that the fields are coherent, it is easy to generalize these results to fields with any level of coherence [3]. The exact conservation along rays of these functions, together with the physical significance of their angular projections, make them potentially useful tools in the description of wave propagation in homogeneous media. One can use these representations in numerically efficient computations of the propagation of partially coherent fields [4].

The current treatment is valid for homogeneous media away from sources. Several generalizations could be considered, however, like the radiance analogues for homogeneous anisotropic or active media. In fact, the determination of the boundary conditions that these functions satisfy at the interface between two homogeneous media would allow us to use this framework for modeling the propagation of partially-coherent vector fields through optical systems of arbitrary numerical aperture.

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REFERENCES