Average coherence approximation for partially coherent optical systems

Boaz Salik, Joseph Rosen, and Amnon Yariv

California Institute of Technology, M.S. 128-95, Pasadena, California 91125

Received January 2, 1996; accepted April 19, 1996; revised manuscript received May 21, 1996

We introduce an approximation to the Hopkins integral for partially coherent optical systems. The average coherence approximation yields results close to the Hopkins integral for a wide range of coherent transfer functions and illumination functions and is far less computationally demanding than the full Hopkins integral.

Monochromatic illumination systems always possess complete spatial coherence.[1] Since all physical illumination sources have a nonzero linewidth, their radiation is more generally described as partially coherent. Thus partially coherent imaging is important in a variety of fields, including astronomy, photolithography, and medicine. Given a coherent impulse response K, propagation through a system (Fig. 1) is described by the Hopkins equation[2]

$$|E_0(x', y')|^2 = \int\int\int\int E(x, y)E^*(\tilde{x}, \tilde{y})J(x, y; \tilde{x}, \tilde{y}) \times K(x, y; x', y') \times K^*(\tilde{x}, \tilde{y}; x', y')\mathrm{d}x\mathrm{d}y\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}, \tag{1}$$

where E is the field, J is the mutual intensity function, (x, y) are the system's input coordinates, and (x', y') are the system's output coordinates. This equation, though general, is tedious to compute, both analytically and numerically; still, it is used almost universally in analysis and simulation of partially coherent systems.[3–5] There have been several attempts to simplify this expression and derive other formulas that are less computationally demanding,[6–9] but they have all addressed special cases of the impulse response or mutual intensity functions. Here we introduce an approximation to the Hopkins equation that is valid for a wide range of coherence functions and impulse responses and considerably reduces the computation time required for determining the output intensity. This approximation is particularly useful when, given J and K, the output intensity is desired for varying input fields, e.g., in iterative routines that optimize the input field to yield a desired output intensity.[3]

For simplicity we begin with one transverse dimension, then generalize to two-dimensional problems. The one-dimensional Hopkins integral is, analogously to Eq. (1),

$$|E_0(x')|^2 = \int\int E(x)E^*(\tilde{x})J(x, \tilde{x})K(x, x') \times K^*(\tilde{x}, x')\mathrm{d}x\mathrm{d}\tilde{x}. \tag{2}$$

Our strategy is to decompose the contribution to $E_0(x')$ by each point $E(x)$ into a coherent component and an incoherent component, then sum the contributions of all points in the x plane to the field intensity at x'. The coherent radiation component interferes coherently with the coherent radiation from all other points x, whereas the incoherent component interferes incoherently with all other radiation. To determine the coherent and incoherent components of radiation from each point, we must somehow average its coherence $J(x, \hat{x})$ with all other points $\hat{x}$ (Fig. 2). This average should be weighted by the contribution of the points to the intensity at x', and this contribution is given by $|K(x, x')|^2$. Thus we can define a function $f(x, x')$ that gives the fraction of the intensity at x that interferes coherently with all other coherent field contributions:

$$f(x, x') = \frac{\int |K(x', \hat{x})|^2 \mu(x, \hat{x})\mathrm{d}\hat{x}}{\int |K(x', \tilde{x})|^2 \mathrm{d}\tilde{x}},$$

$$\mu(x, \hat{x}) = \frac{J(x, \hat{x})}{[J(x, x)J(\hat{x}, \hat{x})]^{1/2}}, \tag{3}$$

where $\hat{x}$ and $\tilde{x}$ are dummy variables at the input plane and the functions K and J are normalized to ensure that $0 \leqslant f \leqslant 1$ (this condition is imposed because f is by definition the fraction of incident power that is spatially coherent).

We are now able to consider the coherent and incoherent contributions to the field at x' independently, using the following relations:

$$|E_0(x')|^2 = |E(x) * K(x, x')|^2, \qquad E(x) \text{ coherent},$$

$$|E_0(x')|^2 = |E(x)|^2 * |K(x, x')|^2, \qquad E(x) \text{ incoherent}, \tag{4}$$

where $f(x) * g(x, x') \equiv \int f(x)g(x, x')\mathrm{d}x$ is a general superposition integral.

0740-3232/96/1002086-05$10.00

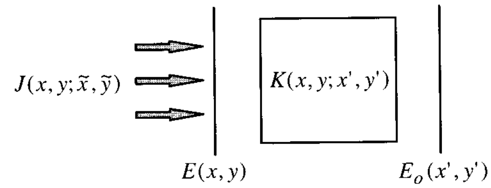

Fig. 1. Schematic diagram of the general optical system described by the Hopkins equation.

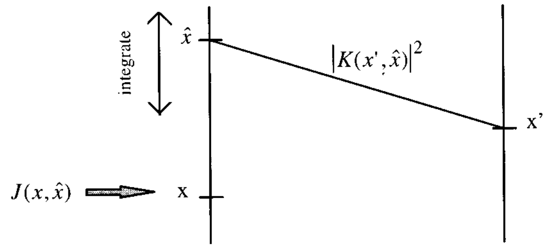

Fig. 2. Schematic representation of the system configuration in the average coherence approximation (x can be an n-dimensional vector, in general).

In our case, $K(x)$ is assumed to have two components, one coherent and one incoherent, i.e.,

$$\begin{aligned}|K(x, x')|^2 &= |K_c(x, x')|^2 + |K_i(x, x')|^2 \\ &= |[f(x, x')]^{1/2}K(x, x')|^2 \\ &\quad + |[1 - f(x, x')]^{1/2}K(x, x')|^2,\end{aligned} \tag{5}$$

so the output intensity is given by

$$\begin{aligned}|E_0(x')|^2 &= |E(x) * K_c(x, x')|^2 \\ &\quad + |E(x)|^2 * |K_i(x, x')|^2 \\ &= E(x) * |[f(x, x')]^{1/2}K(x, x')|^2 \\ &\quad + |E(x)|^2 * |[1 - f(x, x')]^{1/2}K(x, x')|^2\end{aligned} \tag{6}$$

and we have effectively reduced the double integral of the one-dimensional Hopkins equation to a sum of two single integrals. When we use the form of $f(x)$ defined in Eq. (3), Eq. (6) implies that

$$\begin{aligned}|E_0(x')|^2 &= \left| \int E(x)[f(x, x')]^{1/2}K(x, x')\mathrm{d}x \right|^2 \\ &\quad + \int |E(x)|^2[1 - f(x, x')]|K(x, x')|^2\mathrm{d}x.\end{aligned} \tag{7}$$

The two-dimensional generalization is straightforward; Eq. (3) becomes

$$f(x, y, x', y') = \frac{\int\int |K(x', y', \hat{x}, \hat{y})|^2 \mu(x, y, \hat{x}, \hat{y})\mathrm{d}\hat{x}\mathrm{d}\hat{y}}{\int\int |K(x', y', \tilde{x}, \tilde{y})|^2 \mathrm{d}\tilde{x}\mathrm{d}\tilde{y}}, \tag{8}$$

and Eq. (7) is now

$$\begin{aligned}|E_0(x', y')|^2 &= \left| \int\int E(x, y)[f(x, y, x', y')]^{1/2} \right. \\ &\quad \left. \times K(x, y, x', y')\mathrm{d}x\mathrm{d}y \right|^2 \\ &\quad + \int\int |E(x, y)|^2[1 \\ &\quad - f(x, y, x', y')]|K(x, y, x', y')|^2\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{9}$$

Note that, although computing f requires an integration, this need be done only once, given the system parameters K and J, after which computing the output intensity requires only one integration dimension per transverse dimension. Furthermore, if our system is space invariant, i.e., if

$$\begin{aligned}K(x, y, x', y') &= K(x - x', y - y'), \\ J(x, y, \hat{x}, \hat{y}) &= J(x - \hat{x}, y - \hat{y}),\end{aligned} \tag{10}$$

then the integrals of Eqs. (3) and (7)–(9) become convolutions, which are efficiently calculated by use of fast Fourier transforms [because of the space invariance, Eqs. (3) and (7) become one-dimensional convolutions and Eqs. (8) and (9) become two-dimensional convolutions]. Further simplification is possible if we assume specific forms for K and J. For example, if we assume one-dimensional imaging by a rectangular aperture of size D, then

$$K(x, x') = \mathrm{sinc}\left(\frac{x - x'}{d}\right) = \frac{d}{\pi(x - x')} \sin\left[\frac{\pi(x - x')}{d}\right], \tag{11}$$

where $d = \lambda f/D$ is the system's resolution radius, so that

$$|K(x, x')|^2 = \mathrm{sinc}^2\left(\frac{x - x'}{d}\right), \tag{12}$$

and, if our illumination source is rectangular,[8]

$$J(x, \tilde{x}) = \mathrm{sinc}\left(\frac{x - \tilde{x}}{a}\right) = \frac{a}{\pi(x - \tilde{x})} \sin\left[\frac{\pi(x - \tilde{x})}{a}\right], \tag{13}$$

where a is the coherence radius. Thus we have

$$\begin{aligned}f(x, x') &= \frac{1}{\int \mathrm{sinc}^2\left(\frac{\tilde{x} - x}{d}\right)\mathrm{d}\tilde{x}} \int \mathrm{sinc}^2\left(\frac{\tilde{x} - x'}{d}\right) \\ &\quad \times \mathrm{sinc}\left(\frac{\tilde{x} - x}{a}\right)\mathrm{d}\tilde{x}.\end{aligned} \tag{14}$$

Using the substitutions $\hat{x} = \tilde{x} - x'$ and $\bar{x} = x - x'$ simplifies Eq. (14) to

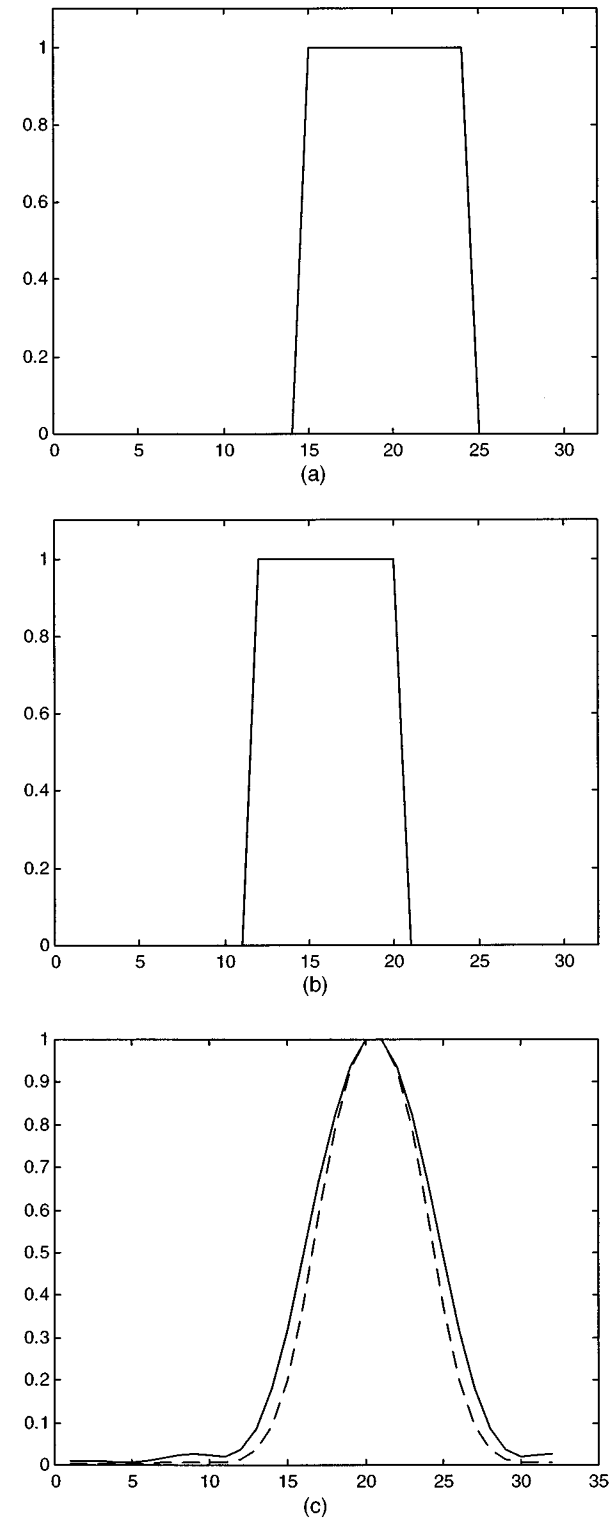

Fig. 3. One-dimensional slit in (a), after imaging through a finite aperture (b), yields the patterns in (c) when either the Hopkins integral (solid curve) or the ACA (dashed curve) is used. Here $J(x, x')$ is a sinc function half as wide as the image.

$$f(x, x') = \frac{1}{\int \operatorname{sinc}^2\left(\frac{\hat{x}}{d}\right)\mathrm{d}\hat{x}} \int \operatorname{sinc}^2\left(\frac{\hat{x}}{d}\right)\operatorname{sinc}\left(\frac{\hat{x} - \bar{x}}{a}\right)\mathrm{d}\hat{x}, \tag{15}$$

which, when we Fourier transform and assume that $d > 2a$, becomes

$$f(x, x') \propto \operatorname{sinc}^2\left(\frac{x - x'}{d}\right). \tag{16}$$

Relation (16) can now be used *a priori* in Eq. (7), eliminating the integration that is due to $f(x, x')$.

To evaluate the error of the average coherence approximation we recall that J is unity for coherent imaging systems and is a delta function for incoherent systems. Therefore (in one transverse dimension) Eq. (7) is identical to a Hopkins integral, where $J(x, \hat{x})$ is replaced by

$$J'(x, \tilde{x}, x') = [f(x, x')f(\tilde{x}, x')]^{1/2} + \{[1 - f(x, x')] \times [1 - f(\tilde{x}, x')]\}^{1/2}\delta(x - \tilde{x}). \tag{17}$$

Therefore the error in field intensity as given by Eq. (7) is

$$\begin{aligned} e(x') = & \int E(x)E^*(\tilde{x})K(x, x)'K^*(\tilde{x}\, x')J(x, \tilde{x})\mathrm{d}x\mathrm{d}\tilde{x} \\ & - \int E(x)E^*(\tilde{x})K(x, x')K^*(\tilde{x}, x') \\ & \times J'(x, \tilde{x}, x')\mathrm{d}x\mathrm{d}\tilde{x} \\ = & \int E(x)E^*(\tilde{x})K(x, x')K^*(\tilde{x}, x')[J(x, \tilde{x}) \\ & - J'(x, \tilde{x}, x')]\mathrm{d}x\mathrm{d}\tilde{x}. \end{aligned} \tag{18}$$

This expression is easier to compute than a direct subtraction of Eq. (7) from Eq. (2), and is used in the examples below.

The error will in general depend on the spatial coordinates, input field, system kernel K, and coherence function J; specifically, the error becomes smaller as $|K(x', x)|$ becomes sharper [its energy is concentrated near some (x', x)]. If our system is an imaging system, for example, then as its numerical aperture becomes larger $|K(x, x')|^2 \rightarrow \delta(x - x')$ and therefore (assuming that J is normalized)

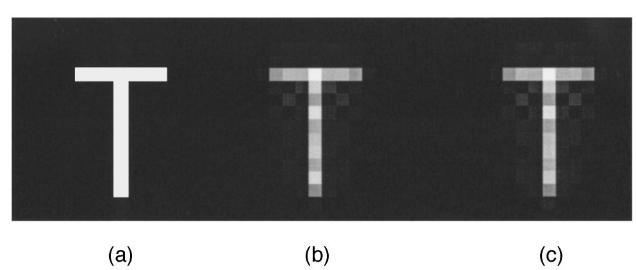

Fig. 4. Pattern in (a) after imaging through a finite aperture becomes (b) under the average coherence approximation and (c) by use of the full Hopkins equation. The aperture size is half of the image bandwidth, and coherence diameter is half of the image size; average error/pixel, 0.95%.

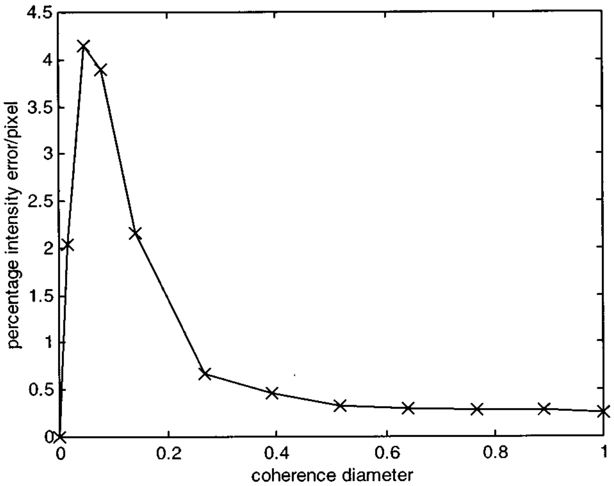

Fig. 5. ACA error versus normalized coherence diameter for a one-dimensional finite aperture undergoing Fresnel diffraction.

$$
\begin{aligned}
f(x, x') &= \int \delta(\hat{x} - x')J(x, \hat{x})\mathrm{d}\hat{x} = J(x, x') \Rightarrow \\
e(x') &= \int\int E(x)E^*(\tilde{x})K(x, x') \\
&\quad \times K^*(\tilde{x}, x')J(x, \tilde{x})\mathrm{d}x\mathrm{d}\tilde{x} \\
&\quad - \left\{ \left| \int E(x)[J(x, x')]^{1/2}K(x, x')\mathrm{d}x \right|^2 \right. \\
&\quad \left. + \int |E(x)|^2 |1 - J(x, x')| |K(x, x')|^2 \mathrm{d}x \right\} \\
&= |E(x')|^2[1 - J(x', x') - 1 + J(x', x')] \\
&= 0.
\end{aligned} \tag{19}
$$

Furthermore, our error goes to zero as we approach full coherence and full incoherence. In the fully coherent case, $J(x, y, x', y') = 1$; thus $f(x, y, x', y') = 1$. From Eq. (6) it is clear then that

$$|E_0(x')|^2 = |E(x')^*K(x')|^2, \tag{20}$$

which is the exact expression for coherent propagation. For the incoherent case,

$$
\begin{aligned}
J(x, x') &= \delta(x - x') \Rightarrow \\
f(x, x') &= \frac{|K(x, x')|^2}{J(x, x)\int |K(x, x')|^2 \mathrm{d}x} = 0,
\end{aligned} \tag{21}
$$

and, again using Eq. (6), we arrive at

$$|E_0(x')|^2 = |E(x')|^{2*}|K(x')|^2, \tag{22}$$

which is the exact expression for incoherent illumination. A notable case in which the error does not approach zero occurs when the impulse response K becomes very wide; then our averaging of the coherence actually loses information, and the results in general may vary from the Hopkins calculation.

Empirically, the average coherence approximation (ACA) yields excellent agreement with the exact Hopkins integral for a wide range of input fields and coherence functions. Figure 3 compares the ACA and the Hopkins integral results for a finite, asymmetric one-dimensional aperture after imaging by a finite aperture (J is a sinc function with a coherence diameter half of the image size). We can see that there is agreement to within 5% of maximum intensity throughout the output plane, and the average error/pixel is 0.15%. Figure 4 compares the output intensities for a two-dimensional input pattern imaged through a finite aperture. Finally, Fig. 5 plots the ACA's average error/pixel (compared to the Hopkins integral) over the output plane for various widths of the coherence function J, assuming Rayleigh diffraction with $z = 16$ pixels. As expected, the error minima lie at full coherence and full incoherence (they are nonzero because of quantization error).

We can estimate the relative computation times of the Hopkins integral and the average coherence approximation by using order-of-magnitude considerations. We assume a grid of n points (i.e., a $j \times k$ two-dimensional grid would have $n = jk$). We can compute the Hopkins integral, assuming a shift-invariant system, by using a triple correlation for the inner integral, $\int E(\tilde{x})J(x, \tilde{x})K^*(\tilde{x}, x')\mathrm{d}\tilde{x}$, which requires $6n^2 \log n + 2n^2 + 2n^2 \log n$ complex operations, assuming that fast Fourier transforms (FFT's) with $O(n \log n)$ are used. The outer integral imposes n iterations of the inner integral plus two complex multiplications, which implies a computational order of $n(8n^2 \log n + 2n^2 + 2n^2) = 4n^3(2 \log n + 1)$. The average coherence approximation requires 2 FFT's, a complex multiplication, and an inverse FFT to compute $f(x, x')$, plus two integrals, each involving 2 FFT's, a complex multiplication, and an inverse FFT. Thus the total computational order is

$$
\begin{aligned}
&2(2n^2 \log n) + n^2 + 2n^2 \log n + 2[2(2n^2 \log n) + n^2 \\
&\quad + 2n^2 \log n] = 3n^2(5 \log n + 1).
\end{aligned}
$$

We can see that the ACA scales approximately with $n^2 \log n$, whereas the Hopkins integral scales as $n^3 \log n$, which for a two-dimensional image is a difference of 2 orders of magnitude. Empirically, the ACA has yielded results 10–50 times faster than the Hopkins integral, depending on the size of the input pattern.

In summary, we have introduced an approximation to the Hopkins integral for propagation of partially coherent fields, that is computationally simpler, both analytically and numerically, and yields good agreement with the Hopkins integral for a wide range of input patterns and system transfer functions. We have presented several analytical conditions for the accuracy of this approximation and numerically analyzed its performance for a range of system parameters. We have shown that the average coherence approximation can be computed significantly faster than the Hopkins integral.

ACKNOWLEDGMENT

This paper is based on research supported in part by an Intel Foundation graduate research fellowship and by grants from the Advanced Research Projects Agency and the U. S. Air Force Office of Scientific Research.

REFERENCES

1. J. W. Goodman, *Introduction to Fourier Optics* (McGraw-Hill, San Francisco, Calif., 1968), p. 106.
2. H. H. Hopkins, "The concept of partial coherence in optics," Proc. R. Soc. London Ser. A **208**, 263–277 (1951).
3. Y. Liu, A. K. Pfau, and A. Zakhor, "Systematic design of phase-shifting masks with extended depth of focus and/or shifted focus plane," IEEE Trans. Semicond. Manuf. **6**, 1–21 (1993).
4. D. C. Cole, E. Barouch, U. Hollerbach, and S. A. Orszag, "Derivation and simulation of higher numerical aperture scalar aerial images," Jpn. J. Appl. Phys. **31**, 4110–4119 (1992).
5. Y. C. Pati and T. Kailath, "Phase-shifting masks for microlithography: automated design and mask requirements," J. Opt. Soc. Am. A **11**, 2438–2452 (1994).
6. J. Perina, *Coherence of Light*, 2nd ed. (Reidel, Dordrecht, The Netherlands, 1985), Chap. 4.
7. L. Mandel, "Concept of cross-spectral purity in coherence theory," J. Opt. Soc. Am. **51**, 1342–1350 (1961).
8. M. Born and E. Wolf, *Principles of Optics* (Pergamon, Oxford, 1965), p. 526.
9. W. H. Carter and E. Wolf, "Coherence and radiometry with quasihomogeneous planar sources," J. Opt. Soc. Am. **67**, 785–796 (1977).