Average Models as Tools for Studying the Dynamics of Switch Mode DC-DC Converters

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Abstract - A novel methodology for teaching the subject of dynamic response of Switch Mode (SM) systems was developed and tested in the class environment. The method applies equivalent circuit models of the power stage and the duty-cycle generation circuitry to describe the low frequency behavior of SM systems and to perform numerical and symbolic analyses by general purpose computer packages (SPICE, MATLAB, MATHEMATICA). Continuous and Discontinuous Conduction Modes (CCM and DCM) of classical PWM topologies, for both voltage and current control methods are discussed.

I. INTRODUCTION

The objective of this paper is to present an approach - adopted by the authors - for teaching the dynamic aspects of Switch Mode (SM) systems in an undergraduate course "Switch Mode DC-DC Conversion" and in graduate studies. The proposed didactical methodology for teaching the dynamic aspects of SM systems, hinges on the Switched Inductor Model (SIM) [1] which is an equivalent circuit representation of the average (low frequency) behavior of the switching action in SM systems.

The present study is unique in that it demonstrates the applicability of Computer Aided Analysis and Design (CAAD) tools for introducing, teaching and investigating the *theoretical* aspects of a SM dynamics. This theoretical 'rebound' is warranted in this case because it is easier to comprehend and study the equivalent circuit representation of an average model than to manipulate matrix representation of SM systems.

The analysis and simulation options, proposed as a vehicle for teaching the dynamics of SM systems, are summarized in the block diagram of Fig. 1. The right arm is the conventional time domain simulation that can be carried out without any special preparations to obtain the cycle-by -cycle response. This can be done by SPICE [2] or any other general purpose electronic circuit oriented software package. The proposed average modeling approach shown on the left arm, can be proceeded by any one of three possible options that can be combined for a comprehensive treatment of a given problem.

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II. THE EQUIVALENT CIRCUIT APPROACH

The equivalent circuit approach for modeling SM systems presented here, hinges on replacing the switching part of the converter by a low frequency, or "average" equivalent circuit and emulating the function of the Duty Cycle modulator.

The Switched Inductor Model (SIM). Close examination of the power stage of common PWM topologies reveals that they all include an inductor which is switched at one end between two points [1]. The switching action is normally carried out by a transistor and a steering diode. The net behavior, however, is that of a switch which toggles the inductor between the two end points (Fig. 2). The Switched Inductor Model (SIM) depicted in Fig. 3 replaces this switching part by an equivalent circuit, using dependent sources, which emulates the average behavior of the three terminals. Hence, the objective of the *Average Equivalent Circuit* approach would be to replace this module by an equivalent circuit, such that the *average* voltages seen across the inductor and the *average* currents flowing through terminals (a), (b) and (c) (Fig. 3), will remain the same as in the physical system. The expressions for the dependent sources, for the general case of continuous and discontinuous conduction modes (CCM and DCM) are as follows [3,4]:

$$G_a = \frac{D_{on}I_L}{D_{on}+D_{off}} \tag{1}$$

$$G_b = \frac{D_{off}I_L}{D_{on}+D_{off}} \tag{2}$$

$$G_c = I_L \tag{3}$$

$$E_L = V_{ac}D_{on} + V_{bc}D_{off} \tag{4}$$

For the CCM case equations (1-4) can be simplified by substituting :

$$D_{off} = 1 - D_{on} \tag{5}$$

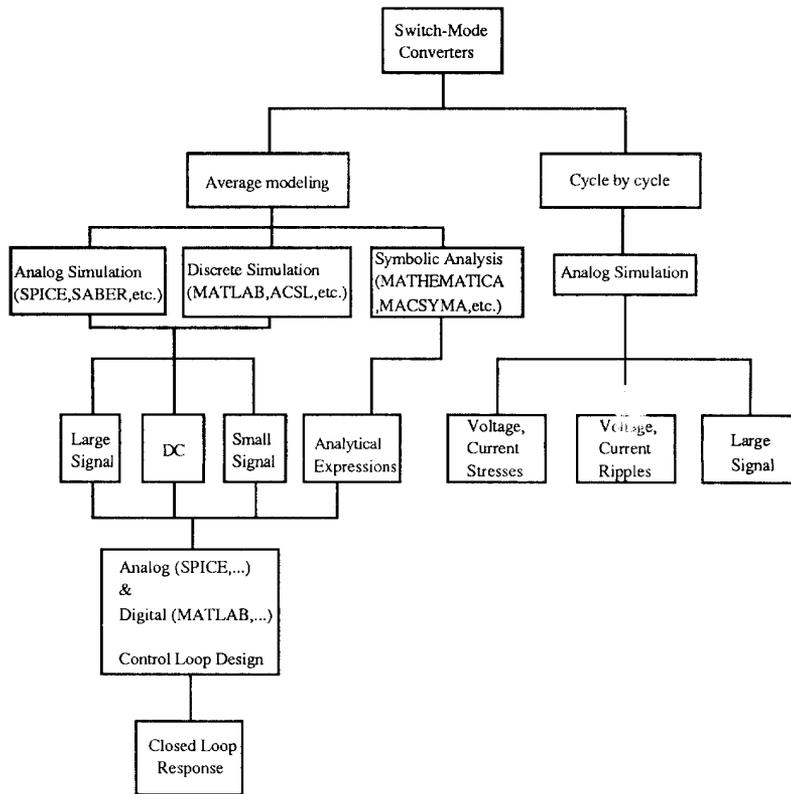and D_{off}, in this case, can be removed from the model.

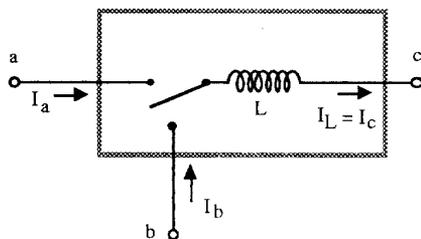Fig. 1   Simulation options of switch-mode converters.
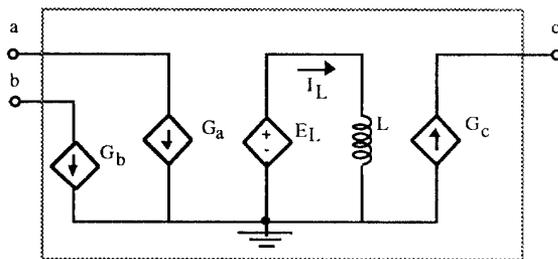

Fig. 2.   The Switched Inductor (SI).

Fig. 3.   The Switched Inductor Model (SIM).

As already shown, the SIM approach can be extended to peak and average current mode [5,6] to quasi resonant converters [7-8] and to PWM based Magamp stabilizers [9]. For the sake of brevity only voltage mode and peak current mode control will be covered here.

The 'D_{on}' Generator. I: Voltage Mode. The dependent sources of the SIM module presented above are a function of the voltages across its ports, the average current of the inductor and the duty cycle (D_{on} and D_{off}). Except for the latter, all other variables can be sensed within the module itself. To operate the SIM, an external excitation of the duty cycle (D_{on} and D_{off}) must be provided.

The equivalent circuit of the 'D_{on}' Generator for the simple PWM case should emulate the basic relationships of a PWM modulator [10]:

$$D_{on} = K_p V_e \tag{6}$$

where K_p is the modulation constant and V_e is the output of the error amplifier (which is the input to the modulator).

The 'D_{on}' Generator. II: Current Mode. Unlike the simple PWM case discussed above, the 'D_{on}' Generator in current mode converters is more complex [10,11]. Generalizing the

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analytical derivations of [11], the duty cycle produced by the current mode generator can be expressed in a topology independent form as (see [10,11] for more details):

$$D_{on} = \frac{V_e - I_L K_s}{V_p + K_s \frac{T}{2L} V_{ac}} \qquad (7)$$

Corrections that take into account sampling effects [12,13] or the drift of the base line [14] can be easily incorporated by modified or additional transfer function placed in a tandem with the basic Duty Cycle Generator. In fact, the equivalent circuit approach presented here can be an excellent vehicle for testing the effects of these suggested corrections.

The 'D$_{off}$' Generator: (for both Voltage and Current Modes). For the case of DCM mode the expression for D$_{off}$ is :

$$D_{off} = \frac{2 I_L L f_s}{V_{ac} D_{on}} - D_{on} \qquad (8)$$

where:

I_L is the average current flowing through the inductor L.
f_s is the switching frequency

In the CCM case, eq. (8) yields [9]:

$$D_{off} \geq 1 - D_{on} \qquad (9)$$

Therefore, by clamping the right side of eq. (8) to (1 - Don) we get the correct D$_{off}$ for both CCM and DCM.

III. ANALOG SIMULATION

Compatibility with general purpose circuit simulator is obtained by replacing the inductor and switches by the SIM equivalent circuit and by defining the Duty Cycle Generators. The rest of the circuit is left as it is. The example of Fig. 4 describes the SPICE compatible equivalent circuit for a buck converter, operating in CCM and DCM modes. Here the switched inductor assembly has been replaced by the SIM model. The 'on' duty cycle (D$_{on}$) is generated by an independent voltage source, for open loop simulations, while the 'off' duty cycle (D$_{off}$) is generated by a controlled voltage source which emulates equation (8). The diode D$_1$ clamps the 'D$_{off}$' generator (E$_{Doff}$) to zero, to prevent negative solutions, while the diode D$_2$ and the controlled voltage source E$_{max}$ clamps it to (1-D$_{on}$) which is reached when the converter enters the continuous conduction mode. A demonstration of the average model behavior relative to the actual operation of the switched circuit is shown in Fig. 5. The results of the average model simulation follow, accurately and smoothly, the average values of the rippled waveforms obtained by a cycle by cycle simulation. The speed up ratio of the average model simulation was found to be more that of 100 folds. The major benefit of the averaging technic is in the ability to linearize the model, which is done automatically by SPICE and to get the small signal transfer functions for frequencies lower than half of the switching frequency.
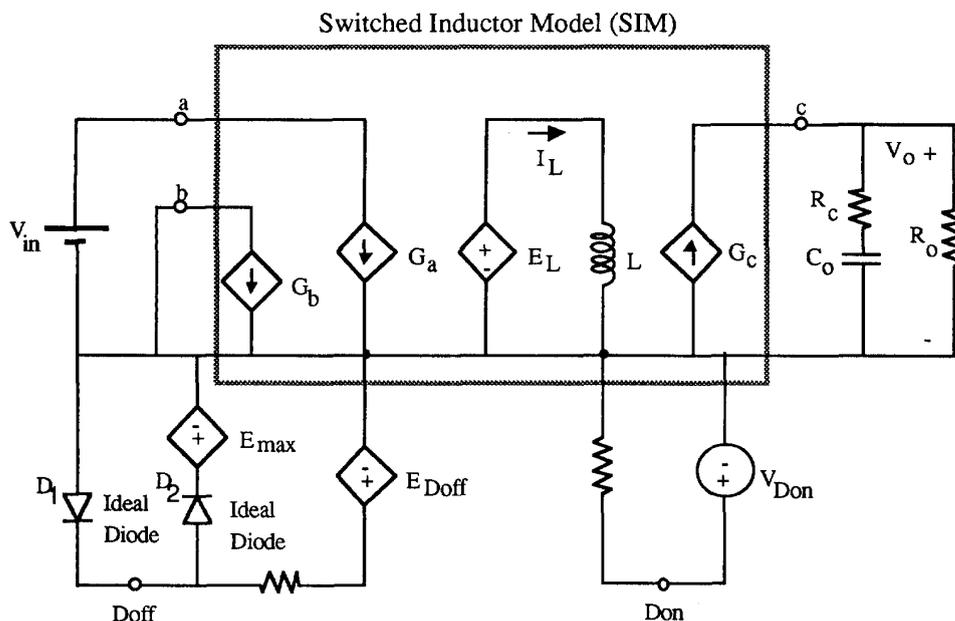
Switched Inductor Model (SIM)

Fig. 4. SPICE compatible average model for opened loop Buck converter (for CCM and DCM).
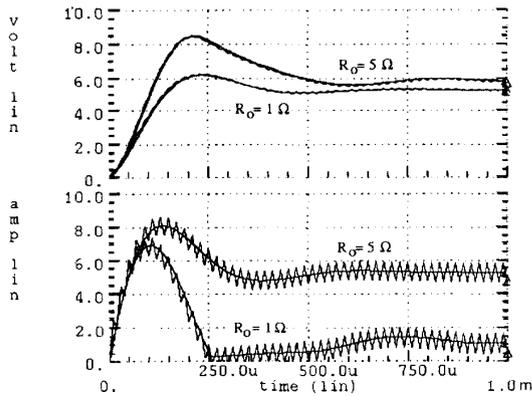
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Fig. 5   SPICE startup response of opened loop buck
converter obtained by   average model
simulation (smoothed line) and cycle by cycle
simulation (rippled line) for two values of
output resistor (R_O). Upper trace: Output
voltage (V_o), Lower trace: Inductor current
(I_L). The buck converter parameters: $V_{in}=12V$,
$L=50\mu H$, $C=100\mu F$, $R_c=0.1\Omega$, $D_{on}=0.5$. (See
Fig. 4 for notations).

IV.   SYMBOLIC ANALYSIS

Notwithstanding the power of numerical simulation, a
solid analytical understanding is still a fundamental
requirement for intelligent analysis and design. In the
approach proposed here, the starting point of the analytical
derivation is the average equivalent circuit rather than the
original circuits. As it turns out, this procedure not only
simplifies the analytical chore but is easy to explain and
comprehend.

The average models of the proposed approach are large
signal equivalent circuits which represent the average (DC and
low frequency) behavior of the switching circuit. The DC
transfer ratio can be derived by applying the fact that at steady
state, the average voltage across the inductor is zero. This
implies that:

$$E_L = 0 \qquad (10)$$

Since:

$$E_L = V_{ac}D_{on} + V_{bc}D_{off} \qquad (11)$$

We get:

$$\frac{V_{cb}}{V_{ac}} = \frac{D_{on}}{D_{off}} \qquad (12)$$

which for the lossless case reduces to the familiar ideal
transfer ratio of the basic power stages. As an example, for a
lossless Boost converter we find that :

$$V_{cb} = V_{in} - V_o \qquad (13)$$
$$V_{ac} = -V_{in} \qquad (14)$$

Substituting into (12) and rearranging we get:

$$\frac{V_o}{V_{in}} = \frac{D_{on}+D_{off}}{D_{off}} \qquad (15)$$

In the continuous conduction mode (CCM), using (5),
equation (15) reduces to:

$$\frac{V_o}{V_{in}} = \frac{1}{D_{off}} \qquad (16)$$

To derive the small signal control to output transfer function,
one has first to linearize the equivalent circuit around the
working point i.e. at a given D_{on}, I_L and V_O. Linearization
can be carried out by taking the differential, similar to the
procedure carried out by SPICE [3]. For example, the non
linear dependent sources, G_a, G_b, G_c and E_L , in CCM case
where $D_{on}+D_{off}=1$, are substituted by linear sources as
follows:

$$d\{G_a\} = d_{on}I_L + D_{on}i_l \qquad (17)$$
$$d\{G_b\} = d_{off}I_L + D_{off}i_l \qquad (18)$$
$$d\{G_c\} = i_l \qquad (19)$$
$$d\{E_L\} = v_{ac}D_{on} + V_{ac}d_{on} \qquad (20)$$

where small letters designate small perturbation around the
working point which is denoted by capital letters.

The average, small signal behavior of the Boost converter,
operating in CCM, can thus be represented by the equivalent
circuit of Fig. 6 in which:

$$Z_{in} = R_{in} + s\,L \qquad (21)$$
$$Z_O = \frac{R_O}{1 + s\,C_O\,R_O} \qquad (22)$$
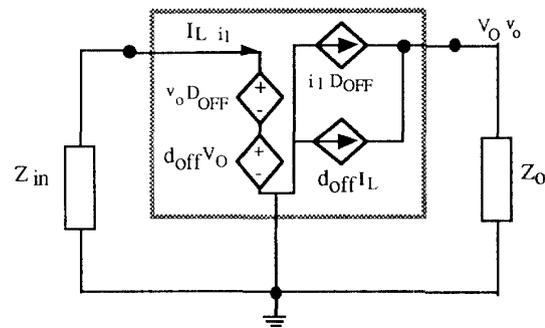
where (s) is the Laplace variable.

Fig. 6   Small signal model of CCM Boost.

It should be noted that in the small signal presentation of
Fig. 6, time dependent variables are denoted by small letters
whereas capital letters represent constants. Also, (d_{off}) in
this case is an independent variable but (v_o) and (i_l) are
dependent variables.

The control to output transfer function (v_o/d_{on}) can now
be derived manually by elementary circuit analysis, by
applying symbolic circuit analysis programs or by general
purpose symbolic programs. APPENDIX demonstrates the

application of the latter (MATHEMATICA, Walfram Research Inc, Champaign, IL). The equivalent circuit of Fig. 6 yields the following expression:

$$\frac{v_o}{d_{on}} = \frac{-R_o(I_L R_{in} + I_L L s - D_{off} V_o)}{R_{in} + (D_{off})^2 R_o + (L + C_o R_{in} R_o)s + C_o L R_o s^2} \quad (23)$$

where:

$D_{off} = 1 - D_{on}$ and hence: $d_{off} = - d_{on}$.

$R_{in} = R_s + R_L$

R_s = switch resistance

R_L = inductor resistance

V_o = steady state output voltage

$I_L = \dfrac{V_o}{D_{off} R_o}$ = steady state inductor current

R_o = output resistance

C_o = output capacitance

L = inductance

For the lossless case, the transfer function reduces to:

$$\frac{v_o}{d_{on}} = \frac{1}{D_{off}} \frac{1 + \dfrac{s}{\omega_z}}{1 + \dfrac{s}{\omega_0 Q} + \left(\dfrac{s}{\omega_0}\right)^2} \quad (24)$$

where:

$$\omega_z = - \frac{R_o (D_{off})^2}{L} \quad (25)$$

$$\omega_0 = \frac{D_{off}}{\sqrt{C_o L}} \quad (26)$$

$$Q = \frac{R_o}{\omega_0 L} \quad (27)$$

which clearly displays the well known characteristics of the Boost control-to-output transfer function [11]: Right Half Plane Zero (RHZ) and a dependence of the resonant frequency and damping on the working point.

V. DISCRETE SIMULATION

Switch mode circuits are in fact non linear control systems. Hence, it would be beneficial to examine their behavior by general purpose software packages which were especially designed for simulating dynamic systems (such as MATLAB, The MathWorks, Inc.). The difference between this approach and that of the analog circuit simulation is two fold:

1. MATLAB and other similar simulators are system rather than circuit simulator. Hence, the system aspects of the problem can be illuminated and better understood when such a tool is applied for studying SM systems.
2. MATLAB type simulators are Discrete Domain simulators. They are extremely powerful tools when applied for studying and designing digital controllers, a task which is incompatible with analog simulators.

In the approach proposed here, discrete simulators are used to examine the average behavior of SM systems rather than the cycle-by-cycle response The advantage of the discrete simulators, such as MATLAB, is their compatibility with discrete control laws including Fuzzy Logic and Neural Networks. This simplifies the exploration of of novel control techniques by being able to remain within the same numerical package. We have used SIMULINK, the graphical tool of MATLAB to define a MATLAB compatible block diagram of the Switched Inductor Model (SIM) as shown in Fig. 7. The model has four input variables: The 'on' duty cycle D_{on} and the voltages at the three terminals V_a, V_b and V_c. It generates three output variables i_a, i_b and i_l ($i_l = i_c$) which are the currents of each of the three terminals of the SIM. The model includes a 'D_{off} generator' that calculates the D_{off} for the DCM and CCM cases. The 'L_f average voltage' block generates the average voltage across the inductor and after integration produces the output variable i_l which represents the time dependent average current of the inductor. The blocks 'Fcn1' and 'Fcn2' emulate the algebraic relationships of eq. (1,2) to generate the output variables i_a and i_b. Once defined, this model can be used to simulate all the basic power stages of switch mode converters.

The diagram of Fig. 8 describes the SIMULINK representation of an open loop Buck converter. The 'SIM' block contains the diagram as described in Fig. 7. The inter-connections of 'vin', 'GND' and the output filter can be changed to get the block-diagrams for Boost and Buck-Boost topologies and their depravities (such as Flyback). Fig. 9 applies the open loop buck diagram of Fig. 8 ('Buck_Aol' in Fig. 9) together with a current mode control block that emulates the relationship described in eq. (7) to generate the 'on' duty cycle (D_{on}). The amplifier and phase compensator that closes the loop can be an analog or digital control law processor. The SIMULINK compatible representation can be used to analyze and run simulations of a SM system under study and can be further explored by any of the available control toolboxes of MATLAB for various types of control design. As an example to MATLAB simulation, the benchmark current mode Buck converter published earlier [11] was analyzed here for various values of current loop gains. The results of the small signal frequency domain analysis (Fig. 10) were found to be in a very good agreement with earlier results which applied other analysis and simulation techniques [11,5].

Fig. 7. SIMULINK (The MathWorks Inc.) compatible Switched Inductor Model (SIM).

Fig. 8. SIMULINK compatible average model for opened loop Buck converter (for CCM and DCM).

VI. RESULTS AND DISCUSSION

The proposed equivalent circuit approach for simulating and studying SM systems has been exposed to the class environment while teaching the undergraduate course "Switched Mode DC-DC Conversion". The SIM model was introduced by an intuitive argument rather than by following the detailed reasoning of a two port network (to be presented in a subsequent paper).

An important area of application of the proposed SIM approach in the classroom is symbolic analysis of SM systems. The advantage here are two fold:
1. The fact that the students are already familiar with the general concept of equivalent circuits and with linear and nonlinear dependent sources.
2. The possibility of applying general purpose symbolic packages to alleviate the chores of the mathematical derivation and manipulation of equations.

The proposed teaching methodology was tested in the class with great success. The ideas and significance of the control-to-output response, output impedance, input impedance (which can be easily shown to represent a negative resistance), audiosusceptibility, current programming and the need for phase compensation and the methods for achieving it, were found to be easy to explain and to comprehend by the students when the proposed numerical and symbolic analyses method was used.
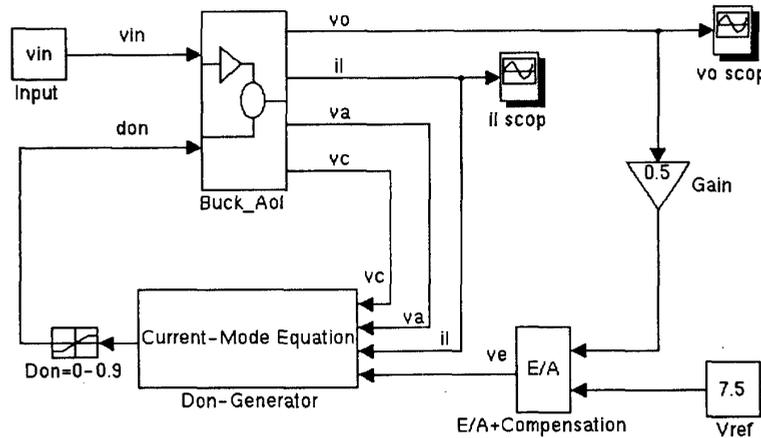

Fig. 9   SIMULINK compatible average model for Current-Mode Buck converter (Closed loop).
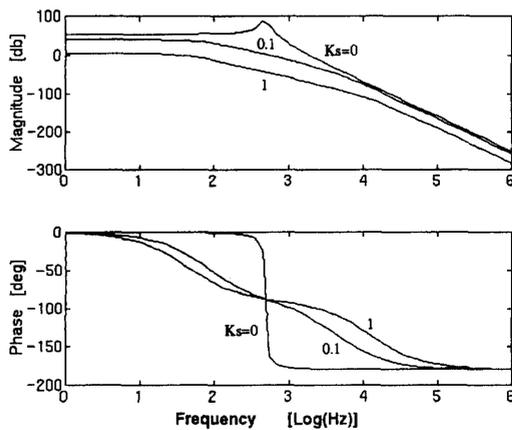

Fig. 10   Magnitude (upper trace) and phase (lower trace) of the small signal , Control (V_e) to Output (V_o) transfer function of the current mode converter shown in Fig. 8 for various values of current gain (K_s). (see detailed parameters of the buck benchmark circuit in [11]).

APPENDIX

(*

MATHEMATICA network analysis session

Since MATHEMATICA accepts only small letters for user defined variables, capital letter notations of the text were changed to small letters. 'DC' values are denoted by double letters (e.g. Vo -> vvo).

1.'DC' solution.

Solving (equations 17 and 18 of the text) for the output voltage and inductor current.
*)

Solve[{vvin==iil rin + ddoff vvo,
vvo == iil ddoff ro}, {vvo, iil}]

$\{\{vvo \to \dfrac{ddoff\ ro\ vvin}{rin + ddoff^2\ ro}, iil \to \dfrac{vvin}{rin + ddoff^2\ ro}\}\}$

(* Getting the transfer ratio (equation 19) *)

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vvo/vvin/.%

$$\frac{ddoff\ ro}{rin + ddoff^2\ ro}$$

(*
2. Small Signal Solution
Defining the rule for parallel connection of impedances.
*)

par[x_,y_] := x y/(x+y)

(* Defining the terminal impedances (Figs 18 and 19). *)
zin = rin + s l
zo = par[ro,1/(s co)]

$$\frac{ro}{co(ro+\dfrac{1}{co\ s})s}$$

(*
Solving for the small signal inductor current (il) and output voltage (vo).
*)

Solve[{vo ddoff + vvo doff + il zin== 0,
iil doff zo + ddoff il zo == vo} , {vo,il}]

{{vo->
$$-\frac{doff\ iil\ ro(-rin - l\ s)}{rin + ddoff^2\ ro + l\ s + co^2\ rin\ ro\ s + co\ l\ ro\ s} -$$

$$\frac{ddoff\ doff\ ro\ vvo}{rin + ddoff^2\ ro + l\ s + co^2\ rin\ ro\ s + co\ l\ ro\ s}\ ,$$

il ->
$$\frac{ddoff\ doff\ iil\ ro}{rin + ddoff^2\ ro + l\ s + co^2\ rin\ ro\ s + co\ l\ ro\ s} -$$

$$\frac{doff\ (1 + co\ ro\ s)\ vvo}{rin + ddoff^2\ ro + l\ s + co^2\ rin\ ro\ s + co\ l\ ro\ s}\}\}$$

Simplify[%]

{{vo->
$$-\frac{doff\ ro\ ((iil\ rin + iil\ l\ s) - ddoff\ vvo)}{rin + ddoff^2\ ro + l\ s + co^2\ rin\ ro\ s + co\ l\ ro\ s}\ ,$$

il ->
$$\frac{doff\ (ddoff\ iil\ ro + vvo + co\ ro\ s\ vvo)}{rin + ddoff^2\ ro + l\ s + co^2\ rin\ ro\ s + co\ l\ ro\ s}\}\}$$

(*
Solving for the small signal control-to-output transfer ratio
(equation 23 in the text).
Note: don =-doff
*)
vo/doff/.%

$$\{\frac{ro\ ((iil\ rin + iil\ l\ s) - ddoff\ vvo)}{rin + ddoff^2\ ro + l\ s + co^2\ rin\ ro\ s + co\ l\ ro\ s}\}$$

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