Modeling and Design Considerations of Coupled Inductor Converters

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Abstract—In this part of the sequel on the modeling and analysis of coupled inductors and coupled inductor based multi-phase switching converters, the recently developed symmetrical coupled inductor model is first extended to include the inductor winding dc resistance (DCR). The extended model is then used to analyze the influence of the coupling on the DCR based current sensing schemes popularly used in multi-phase switching regulators. It is found that the time-constant matching condition in coupled inductor converters needs to be modified to include the coupling coefficient. The proposed model is also used to derive the small-signal control-to-output transfer function of the converters incorporating coupled inductors, with which the effect of coupling on the dynamic behaviors of the converter power stage, such as resonant frequency and damping factor, can be easily evaluated.

I. INTRODUCTION

Coupled inductor as a special form of multiple winding coupled magnetic structure, or transformer, has been around since the early years of electrical engineering. In modern high-frequency switching converters, it has been used in many applications such as multi-output, cross-regulated converters [1, 4], ripple cancellation and multiple magnetic component integration [2, 6], snubbing [3], and transformer/inductor integration [5], etc. More recently, it also found applications in multi-phase buck regulators [7-18]. Moreover, the basic coupled magnetic structure was extended to practical implementations beyond three phases [10-13]. It is generally accepted that reverse coupling between the coupled windings is the preferred coupling scheme [7, 10], and fast transient response can be achieved especially when the control scheme allows multiple phases to overlap in case of fast load transient [20]. Some also made plausible argument that coupled inductor based converter also improves conversion efficiency, but this seems to be valid only under the assumption that in order to achieve the same transient response, a coupled-inductor based converter needs to switch only at a reduced switching frequency. It is noted that the phase current ripple cancelling effect at the output of a regular multi-phase switching regulator is lost in the coupled-inductor based one, leading to higher ripple voltage at the output [20].

Most of the modeling and analysis works on coupled inductors were based on the well-established transformer models, such as the T- or π-equivalent model [10, 14, 15, 19], and the cantilever model [6]. Although these models are largely valid, they are asymmetrical relative to the roughly symmetrical magnetic structure largely used in multiphase buck regulators because the magnetizing inductance appears only in one of the coupled windings. The asymmetry in the model usually makes the analysis very cumbersome. An equivalent inductance model based on the concept of self and mutual inductances in basic circuit theory was introduced in [7, 8], but it still fell short in crystallizing the general relationship involved in coupled inductors. With the understanding that coupled inductors are magnetic structures with loose coupling and that the coupling coefficient plays an important role in the resulting circuit performance, a general symmetrical and invariant model for multi-winding coupled inductors was recently derived [20]. It was based on the same self- and mutual-inductance concept as in [7, 8], but has the advantage in analyzing all the key circuit behaviors of the coupled inductor based multi-phase regulators, such as steady-state ripple currents, and transient response, etc. The model can be easily extended to multiple (more than two) winding cases, and all the circuit parameters involved in the model, such as leakage inductances and the coupling coefficient, can be easily determined based on terminal inductance measurements.

In this paper, the symmetrical model presented in [20] is first extended to include the winding dc resistance (DCR). It is then used to analyze the influence of the coupling on the DCR based current sensing schemes and to derive the small-signal control-to-output model of the converters incorporating coupled inductors.

II. REVIEW AND EXTENSION OF COUPLED INDUCTOR MODELS

Fig. 1 shows the well-known general transformer model of two inversely coupled inductors. It has four parameters: magnetizing inductor $L_m$, turns ratio $n$, leakage inductors $L_{k1}$ and $L_{k2}$, with only three of them independent.

Since this model is asymmetrical in structure, it is unable to provide enough analytical insight into how the coupling coefficient affects a converter’s steady state and transient performances. A new symmetrical transformer model for coupled inductors was developed in [20]. While it was discussed extensively in [20] that the symmetrical model is easy to use and suitable for analyzing converters utilizing coupled inductors, some practical design considerations in coupled inductor converters were not discussed, such as lossless current sensing based on inductor DCR [21, 22], and...
the effect of coupling coefficient on the small-signal model of the converter. In this section, a symmetrical transformer model with non-zero winding resistances will be derived first, which will serve as the basis for the discussions in the subsequent sections.

For simplicity, two inversely coupled identical windings, each with a self inductance of \( L \) and a DCR of \( R_L \), as shown in Fig. 2(a), are assumed. The model can be analytically derived from the following expression governing two inversely coupled windings:

\[
\begin{align*}
\frac{v_{L1}}{L} & = \frac{di_1}{dt} - M \frac{di_2}{dt} + R_L i_1, \\
\frac{v_{L2}}{L} & = -M \frac{di_1}{dt} + L \frac{di_2}{dt} + R_L i_2,
\end{align*}
\]

where \( M \) is the mutual inductance. The expression can also be re-arranged as:

\[
\begin{align*}
\frac{v_{L1}}{L} & = L_k \frac{di_1}{dt} - k v_{L2} + R(i_1 + k i_2) \\
\frac{v_{L2}}{L} & = -k v_{L1} + L_k \frac{di_2}{dt} + R(k i_1 + i_2)
\end{align*}
\]

where \( L_k \) is the leakage inductance defined as \( L_k=(1-k^2)L \) and \( k \) is the coupling coefficient defined as \( k=M/L \).

From (2), one can easily come up with a new symmetrical equivalent circuit as shown in Fig. 2(b).

It can be seen that the model shown in Fig. 2(b) takes on a quite different form from the traditional transformer models where \( R_L \) is simply a series resistor on both the primary and secondary sides of a transformer. The magnetizing inductor, which leads to asymmetry in other model, does not explicitly appear in the new model. It should also be pointed out that \( L_k \) in Fig. 2(b) is the inductance that is directly measurable and can be obtained by measuring across one winding while the other winding is shorted. It is different from the leakage inductances, \( L_{k1} \) and \( L_{k2} \), shown in Fig. 1.

One of the distinctive features of the new equivalent circuit model for two coupled inductors given in Fig. 2(b) is that it is very convenient to derive the dynamic models of converters utilizing coupled inductors. The condition for DCR based current sensing can also be easily determined according to (2) and Fig. 2(b).

III. APPLICATIONS OF THE DERIVED MODEL

3.1 Inductor DCR Based Current Sensing in Coupled Inductor Converters

Lossless inductor DCR based current sensing has become a popular method in multiphase, high current switching regulators such as those for CPU, memory and graphics [21, 22]. The condition to accurately extract inductor current information is well understood in converters with discrete, or uncoupled, inductors. However, such a condition, which is of practical engineering values, is not well established in converters utilizing coupled inductors.

Fig. 3(a) shows the general lossless inductor DCR current sensing circuit in a two-phase interleaved buck converter with discrete inductors. Time-domain waveforms of the phase 1 inductor current, \( i_1(t) \), and the sensed voltage across the capacitor, \( v_{s1}(t) \), are given in Fig. 3(b). It is well-known that with proper design, i.e. when the time constant matches between the inductor and the current sensing \( r \), C network, or \( rC=L/R_L \), the inductor current waveform can be extracted from the voltage waveform across the sensing capacitor, i.e. \( v_{s1}(t) \approx i_1(t)\times R_L \).

A two-phase buck converter circuit using the lossless inductor DCR current sensing circuit with inductors coupled to each other is also shown in Fig. 4 (a), together with the time-domain inductor winding current waveforms [20], \( i_1(t) \) and \( i_2(t) \), as well as voltage waveforms across the sensing
capacitors, $v_{C1}(t)$ and $v_{C2}(t)$, in Fig. 4(b). It is obvious that the sensed capacitor voltage is no longer a replica of the winding current in coupled inductors, no matter how the time constants with the inductor and the $r$, $C$ network are selected, as the frequency of inductor current is twice the frequency of the voltages across the sensing capacitors.

In order to extend the lossless inductor DCR current sensing technique to applications with coupled inductors, the following analysis is carried out in $s$-domain. According to (2),

$$\frac{s}{s+RkL}-\frac{1}{s} = \frac{1}{s+RkL} + \frac{1}{s} \left(1 - \frac{1}{s+RkL}\right), \quad (6)$$

then (5) becomes

$$V_{C1}(s) + V_{C2}(s) = R_L \left[ I_1(s) + I_2(s) \right], \quad (7)$$

or,

$$v_{C1}(t) + v_{C2}(t) = R_L \left[ i_1(t) + i_2(t) \right]. \quad (8)$$

Figure 3. Lossless inductor current sensing with uncoupled inductors: (a) circuit diagram; (b) inductor current and capacitor voltage waveforms.

$$V_{L1}(s) = sL_k I_1(s) - kV_{L2}(s) + R_L \left[ I_1(s) + kI_2(s) \right], \quad (3)$$

and from Fig. 4(a),

$$V_{C1}(s) = \frac{V_{L1}(s)}{1 + sRkC}, \quad V_{C2}(s) = \frac{V_{L2}(s)}{1 + sRkC}. \quad (4)$$

Hence, the sum of the voltages across the sensing capacitors can be obtained as

$$V_{C1}(s) + V_{C2}(s) = R_L \frac{I_1(s) + I_2(s)}{1 + sRkC} \left[1 + s \left(\frac{1}{s+RkL(1+k)}\right)\right]. \quad (5)$$

In the equation above, if $r$ and $C$ are selected such that

$$rC = \frac{L_k}{R_L \left(1 + k\right)} = \frac{(1-k)L}{R_L},$$

Equation (6) is the new time constant matching condition for DCR based current sensing in coupled inductor converters, which differs from that in uncoupled inductor converters. It is apparent that the coupling coefficient has a significant impact on the selection of the $r$ and $C$ values. (7) and (8) show that once (6) is satisfied, the sum of the voltages across the two sensing capacitors is the exact replica of the sum of the currents in both windings.

Figure 4. Lossless inductor current sensing with coupled inductors: (a) circuit diagram; (b) inductor current and capacitor voltage waveforms.
The conclusion expressed in (7) and (8) above, was also discovered in [16, 19]. The difference is that the time constant matching condition obtained in (6) includes the effect of the coupling coefficient, which is usually not close to 1 in coupled inductors, and a clear definition of the leakage inductance given in [20] and (2), and directly measurable with no need to convert or partition.

It should be pointed out that even though the sensed capacitor voltage does not represent the individual inductor winding current in coupled inductors, the result obtained from (6), (7) and (8) is still of significant practical importance. In many applications, obtaining the total current information is critical to implement output voltage positioning, or load-line, and over-current protection.

3.2 Dynamic Modeling of Coupled Inductor Converters

Figure 5 shows the equivalent circuit of a two-phase coupled inductor buck converter where the coupled inductors are replaced by the model given in Fig. 2(b). If we denote \( v_{p1} \) and \( v_{p2} \) as the switching node or phase node voltages (referred to ground), and \( v_{x1} \) and \( v_{x2} \) as the voltages (referred to ground) at the fictitious nodes \( x_1 \) and \( x_2 \), then from Fig. 5, we have,

\[
v_{l1} = v_{p1} - V_0
\]
\[
v_{l2} = v_{p2} - V_0 ,
\]

and,

\[
v_{x1} = v_{p1} + k v_{l2} = (v_{p1} + k v_{p2}) - k V_0
\]
\[
v_{x2} = v_{p2} + k v_{l1} = (k v_{p1} + v_{p2}) - k V_0 .
\]

In order to derive the dynamic model and the control-to-output small signal transfer function of the converter in Fig. 5, averaging modeling approach [23] is adopted in the following discussion. According to (10), the average phase node voltage, \( v_{x} \), can be obtained as

\[
\bar{v}_x = \bar{v}_{x1} = \bar{v}_{x2} = d(l+k)\bar{v}_{in} - k\bar{v}_0 ,
\]

where the symbol "\( \bar{\cdot} \)" is used to denote the average of the corresponding variable. Fig. 6(a) shows the same equivalent circuit model of the converter in Fig. 5, while its equivalent averaging circuit is given in Fig. 6(b).

Assuming \( d = D + \dot{d}(t) \), \( \bar{v}_0 = V_0 + \ddot{v}_0(t) \) and \( \bar{v}_{in} = V_{in} \), where \( \dot{d}(t) \) is the small-signal duty cycle perturbation and \( \ddot{v}_0(t) \) is the resulting perturbation on the output voltage, then the control-to-output transfer function of the power stage, \( G_d(s) \), can be derived as

\[
G_d(s) = \frac{V_0(s)}{d(s)} = \frac{R_0 V_{in}}{R_0 + R_C \frac{L_k}{2} C_0 (s^2 + 2 \omega_0 s + \omega_0^2)} \text{,}
\]

where

\[
\omega_0 \Delta \sqrt{(1+k) \frac{R_0 + R_L/2}{R_0 + R_C} / \frac{L_k}{2} C_0} , \text{ and}
\]

\[
\frac{L_k}{2(R_0 + R_C)} + (1+k)(R_0 // R_C + \frac{R_L}{2})C_0
\]

\[
\frac{2 \Delta \omega_0}{a_0 L_k C_0}
\]

are the resonant frequency and damping factor, respectively.

If \( R_0 >> R_C \) and \( R_0 >> R_L \), which is the case in most applications, (12) can be simplified as

\[
G_d(s) = \frac{V_0(s)}{d(s)} = \frac{V_{in} (1+k)(1+sC_0R_C)}{s^2 \frac{L_k}{2} C_0 + s \left( \frac{L_k}{2R_0} + (1+k)(R_0 // R_C + \frac{R_L}{2})C_0 \right) + (1+k)} \text{,}
\]

and the resonant frequency of the converter power stage transfer function can be simplified as

\[
\omega_0 = \sqrt{\frac{2(1+k)}{L_k C_0} = \sqrt{\frac{2}{(1-k)L_C} } \text{.}
\]

For comparison purposes, the resonant frequency of a two-phase buck converter with the inductors uncoupled \((k=0)\), \( \omega_{dis} \), is

\[
\omega_{dis} = \sqrt{\frac{2}{L_{dis} C_0} \text{,}
\]

where \( L_{dis} \) is the inductance of the uncoupled inductors.
In order to compare the multi-phase power converters with and without the inductors coupled, the following two different design cases can be considered: a) design based on equal magnetic component size, and b) design based on equal transient respond speed. The former implies that $L_{\text{dis}}$ of the uncoupled inductor equals the self inductance of each winding in the coupled one, or $L_{\text{dis}} = L$, while the latter means $L_{\text{dis}}$ equals the leakage inductance of the coupled inductor, or $L_{\text{dis}} = L_k$.

From (14) and (15), it is observed that, regardless of $L_{\text{dis}} = L_k$ or $L_{\text{dis}} = L$, there exists:

$$\omega_0 > \omega_{\text{dis}}$$

for any $k > 0$.

Equation (16) means that, due to coupling, the resonant frequency in converters with coupled inductors is always greater than that in converters with uncoupled inductors, regardless of $L_{\text{dis}} = L_k$ or $L_{\text{dis}} = L$. This conclusion implies that it is possible to design a control loop with higher crossover frequency, or bandwidth, in a regulator utilizing coupled inductors.

IV. SIMULATION AND EXPERIMENTAL VERIFICATION

The DCR based total current sensing scheme in coupled inductor converters as discussed in the previous section was verified through PSpice simulation. The circuit parameters used in the simulation, referred to Fig. 4(a), are listed as follows:

- Input voltage: $V_{\text{in}} = 12$ V
- Output voltage: $V_o = 1.2$ V
- Output current: $I = 20$ A
- Self inductance: $L = 1$ µH
- Coupling coefficient: $k = 0.6$
- Leakage inductance: $L_k = 0.64$ µH
- Inductor DCR: $R_L = 1$ mΩ
- Sensing network parameter: $r = 4$ kΩ, $C = 0.1$ µF
- Per-phase switching frequency: $f_s = 250$ kHz

In this case, the sensing time-constant, $rC = 0.4$ ms, was chosen according to (6) to match the time constant of the coupled inductor, $L/R_L(1+k) = 0.4$ ms. The simulated results of the individual phase currents, $i_1$ and $i_2$, and summed phase current, $i$, sensing capacitor voltages, $v_{C1}$ and $v_{C2}$, and their sum, $v_C$, are shown in Fig. 7(a) in steady-state operation, and in Fig. 7(b) during load transient, respectively. It is obvious that the sum of the voltages on the individual sensing capacitors replicates the summed phase currents exactly in both situations, and the scaling factor is $R_L = 1$ mΩ.

![Figure 7](image-url)

The same test board as used in [20] was used to measure the power converter control-to-output transfer function. The controller used on the test board shown in Fig. 8 is ISL6266, which utilizes the so-called Robust Ripple Regulator (R³) modulator. According to the controller manufacturer [24], the small signal transfer function, or modulation gain, of the modulator, $G_m(s)$, is given by,
\[
G_m(s) = \frac{\dot{d}(s)}{V_{\text{comp}}(s)} = \frac{2}{1.5 \times 10^5 (V_m - V_0) + 3.2V_m \times 10^5 f_s} + \frac{1.067s + 10^5}{f_s}, \quad (14)
\]

where \( V_{\text{comp}}(s) \) is the error compensator output and \( f_s \) is the per-phase switching frequency of the converter.

The coupled inductor is LC1740-R30R09A from NEC/Tokin, which specifies a typical self inductance of 310 nH. The measured parameters (with a short external wire connected to one terminal of each winding to fit a current probe) are \( L = 353.5 \text{ nH} \), \( k = 0.622 \), and \( L_k = 216.7 \text{ nH} \).

It can be seen in Fig. 9 that the low-frequency gain and phase between the predicted and measured results match well, while the measured resonant frequency, \( \omega_0 \), is slightly higher than predicted, which is most probably the outcome of the capacitance reduction effect under output bias voltage. The actual capacitance of the high-value X5R rated MLCC output capacitors is usually reduced by at least 25% under dc bias voltage. With the inclusion of the equivalent parasitic on-board loop resistance in the model, which was actually measured with the injection of a current source, the resonant peak also matches well. Beyond \( \omega_0 \), the predicted and measured results follow the same general trend, but the measured results roll off at a higher frequency, which is partly due to the same capacitance reduction phenomenon and could be partly attributed to the inaccuracy involved in the transfer function of the complex modulator used in the controller.

Based on the information above, the load resistance \( R_0 = 57.5 \text{ m}\Omega \). The test board uses all ceramic capacitors in parallel as the output capacitor. The equivalent ESR \( R_c \) is very low. Since \( R_0 \gg R_c \), and \( R_0 \gg R_{L_1} \), (13) was adopted in the calculation of the Bode-plot in Fig. 9.

Fig. 10 also shows the Bode-plots of the converter power stage control-to-output transfer functions, \( G_d(s) \), with and without the inductors coupled using the same parameters given above. The gain peaks at \( \sim 21.3 \text{ kHz} \) with coupled inductors. With discrete inductors, it peaks at \( \sim 16.1 \text{ kHz} \) when \( L_{dis} = L_k = 216.7 \text{ nH} \) and at \( \sim 12.5 \text{ kHz} \) when \( L_{dis} = L = 353.5 \text{ nH} \), both of which is lower than that with the coupled one. It confirms the prediction made in (16).
V. CONCLUSIONS

In this paper, the previously developed symmetrical coupled inductor model is first extended to include the DCR. The extended model is then used to analyze the influence of the coupling coefficient on the DCR based current sensing schemes popularly used in multi-phase switching regulators. It is found that the well-known time-constant matching condition in uncoupled inductor converters should be modified to include the effect of $k$ when inductors are coupled, and only the summation of the sensed phase currents is valid. Finally, the proposed coupled inductor model is also used to derive the small-signal control-to-output model of the converters incorporating coupled inductors. The derivation shows that the resonant frequency of the power stage is effectively increased through coupling, indicating in practical applications a higher bandwidth loop design is achievable in regulators incorporating coupled inductors.

REFERENCES


