Positive-Feedback Oscillators

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***Abstract*—These lecture notes describe the basic principles of positive-feedback sinusoidal oscillators.**

***Index Terms*—Barkhausen criterion, Laplace transform, natural frequencies, natural response, Nyquist stability criterion, passive network, positive-feedback linear sinewave oscillators, transient response, zero-input response.**

I. Introduction

OUR aim is to develop an electronic circuit that is able to produce *sustained* sinusoidal oscillations at a *single* frequency. No external ac source will drive the circuit; it should start from an **initial state**, and its **zero-input response** should approach a sinewave. We assume that in steady state, the circuit that we develop can be described by a linear, lumped, time-invariant (LLTI) small-signal equivalent model. We shall call such a circuit **linear sine-wave oscillator**.

A. Terminology

Before we start the development of an oscillator, let us pay some attention to the terminology accepted in basic circuit theory [1].

We recall that the zero-input response of an LLTI circuit is also called the **natural response**. It is the response of the circuit to an initial state only, with all the independent sources suppressed.

The response of the circuit to an **external excitation** only, with the initial state equals zero, is called **zero-state** or **forced response.**

The **complete response** of the circuit is a superposition of the natural and the forced responses. The complete response of a stable circuit can be also divided into the **transient response** and the **steady-state response.**

The circuit transient response is contributed by both the initial state and the external excitation. For a **stable circuit**, the transient response is decaying and the **circuit steady state** depends on the external excitation only.

Note, that the oscillator that we develop should behave in a different manner compared to a stable circuit: its forced response is zero, and its steady state should be defined by its natural response. Hence, the oscillator natural response should not decay with time. Instead, it should approach a sinewave at a desired frequency and amplitude.

B. Applications

In terms of applications, oscillators, either sinusoidal or not, single or multi-frequency, linear or nonlinear, are generally used to synchronize a system (computers), to carry and detect information (communication), and to test a system (instrumentation).

C. Principles of Operation

Depending on the principle of operation, oscillators can be classified as **negative resistance**, **parametric**, **relaxation**, and **positive-feedback**. We shall treat here the latter type only.

II. Positive-Feedback Oscillators

Positive-feedback oscillators employ an LLTI **passive frequency-selective** electric network (**feedback network**) $\beta(j\omega)$ and an **active, electronic amplifier** A_{OL} interconnected in a **positive-feedback loop**. We shall assume here that A_{OL} is frequency-independent.

A. Passive Feedback Network

Recall that a passive electric network is that combined of **passive elements**: resistors, capacitors, and inductors. A **passive resistor** never dissipates and a **passive capacitor and inductor** never store **negative energy** [1]. A circuit or its elements are active if they are not passive.

It is quite clear that being *always* stable, a passive network alone is not able to produce sustained oscillations: it has no poles in the right half of the **Laplace plane**, and, therefore, its natural response approaches zero.

To illustrate this, we recall that the **transfer function** of a network in terms of the **Laplace transform** [1],

$$\boldsymbol{L}[f(t)] \equiv \int_{0_-}^{\infty} f(t)e^{-st}dt, \tag{1}$$

where s is the **complex frequency**, $s=\sigma+j\omega$, is defined as follows [1]:

$$H(s) \equiv \frac{\boldsymbol{L}[\text{zero}-\text{state response}]}{\boldsymbol{L}[\text{input}]} = K\frac{\prod_{j=1}^{m}(s-z_j)}{\prod_{i=1}^{n}(s-p_i)}, \tag{2}$$

where K is a constant, the z_j's are called **zeros**, since $H(z_j)=0$, and the p_i's are called **poles**, since $H(p_i)=\infty$, provided that $z_j \neq p_i$.

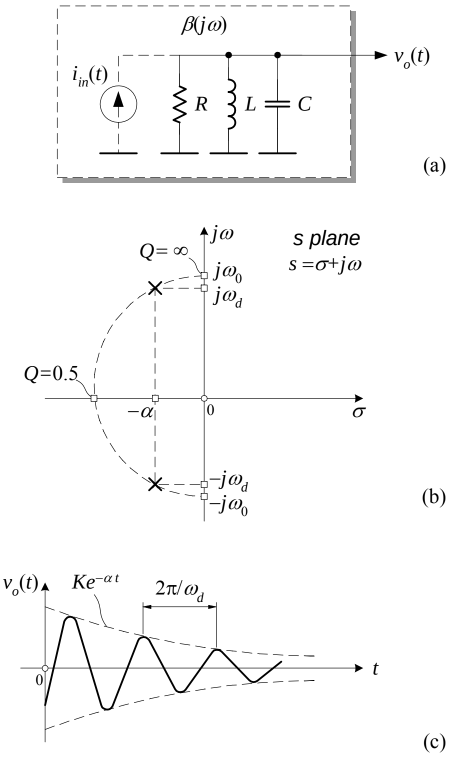

Fig. 1. A parallel *LCR* network (a), locus of its natural frequencies (b), and its natural time response (c). The characteristic equation: $s^2+2\alpha s+\omega_0^2= s^2+(\omega_0/Q)s+\omega_0^2$. The resonant frequency $\varpi_0 = 1/\sqrt{LC}$, *L*, and *C* are kept constant, and *R* varies. $Q=R/(\omega_0 L)$. Since *R* is never equal infinity, the time response is always decaying. $H(j\omega_0)$ never equals infinity.

The **partial-fraction extension** of (2) can be written as

$$H(s)=\sum_{i=1}^{n} K_i \frac{1}{s-p_i}, \tag{3}$$

where the K_i's are called the **residues** of the particular poles [1].

Applying **the inverse Laplace transform** $\boldsymbol{L}^{-1}[H(s)]$ to (3)—this is usually done with the help of a look up table—one can obtain the natural response of the network,

$$\text{natural response} = \sum_{i=1}^{n} K_i e^{p_i t}, \tag{4}$$

where the poles p_i's are also called **natural frequencies** (or **natural modes** [2]) of the network. The p_i's in (4) depend on the network topology, and the K_i's (possibly complex) depend, in addition, on the initial state.

Recall that a **network function** (a particular transfer function from an input variable to an output variable) and the natural frequencies of a network can be determined by applying an excitation that does not change the natural structure of the circuit [1], [2]. This can be done either with an independent current source connected in parallel to a network branch, or with an independent voltage source

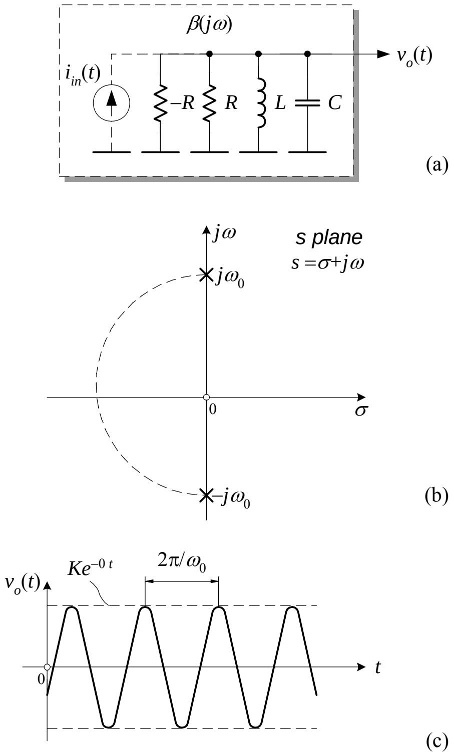

Fig. 2. A parallel *LCR* network employing negative-resistance (a), its natural frequencies (b), and its natural time response (c). Since $R\|(-R)=\infty$, $H(j\omega_0)$ *does* equal infinity.

connected in series with a network branch. As far as the natural response is concerned, the above independent sources turn into an open or closed circuit, correspondingly; thus, not altering the natural structure of the network.

Consider for example a passive *LCR* network of Fig. 1(a). Its network function,

$$\beta(s)=\frac{\boldsymbol{L}[v_o(t)]}{\boldsymbol{L}[i_{in}(t)]}\equiv Z(s)=\frac{1}{C}\frac{s}{(s-p)(s-p^*)},$$

$$p=-\alpha+j\varpi_d,\ p^*=-\alpha-j\varpi_d \qquad , \tag{5}$$

$$\alpha=\frac{1}{2RC},\ \varpi_0=\frac{1}{\sqrt{LC}},\ \varpi_d\equiv\sqrt{\varpi_0^2-\alpha^2}$$

can be found by connecting an independent current source $i_{in}(t)$, as shown in Fig. 1(a).

Locus of the natural frequencies for the *LCR* network is shown in Fig. 1(b). This figure and (5) clearly demonstrate that the poles of a practical network (with a finite and positive *R* and, hence, α) are always located to the left of the $j\omega$ axis. Or in other words, the denominator of (5) is never zero for any *s* with a nonnegative σ. As a result, a practical *LCR* network cannot generate a sustained sinewave. Its output is always decaying, as shown in Fig. 1(c), and its natural response approaches zero.

One way to reach nondecaying oscillations is to compensate for the network positive (passive) resistance R (read energy losses) with an element having negative (active) resistance $-R$ (read energy pumping into the circuit), as shown in Fig. 2(a).

Negative-resistance oscillators are based just on this idea. They employ nonlinear electronic elements, such as the tunnel diode, having negative slope in a part of their voltage-current characteristic, to obtain—for relatively small signals—negative resistance.

Negative-resistance approach to oscillators is generally used at radio (RF) and especially at microwave frequencies, where it becomes difficult to construct a feedback circuit without introducing excess phase shift.

At relatively low frequencies, from kilohertz up to hundreds of megahertz, positive-feedback approach is generally used.

B. Feedback Loop

Another way to make the above *LCR* network to oscillate continuously is to integrate it in a feedback loop, as shown in Fig. 3(a), such that the total network function becomes

$$H(s) = \frac{S_o}{S_{in}} = \frac{A_{OL}}{1 - A_{OL}\beta(s)} = K\,\frac{P(s)}{Q(s)}\,. \tag{6}$$

Note that contrary to (5), the denominator of (6) *can* reach zero for an s_1 with a non-negative σ, namely, it happens when the loop gain $A_{OL}\beta(s_1)=1$.

C. Barkhausen criterion

To reach sustained oscillations in steady state, the loop gain $A_{OL}\beta(s_1)$ should be equal to unity at a physical, ω—not complex, s—frequency, namely,

$$A_{OL1}\beta(j\omega_1) = 1 \Rightarrow \begin{cases} \left|A_{OL1}\beta(j\omega_1)\right| = 1, \text{amplitude criterion} \\ \varphi_{A_{OL1}\beta(j\omega_1)} = 0, \text{ phase criterion} \end{cases}. \tag{7}$$

where $A_{OL1}\beta(j\varpi_1)$ and ω_1 are the oscillator's steady-state loop gain and frequency; index 1 emphasizes that in steady state the loop gain $A_{OL1}\beta(j\omega_1)$ is unity.

Since for a physical frequency, σ=0, the oscillator's steady-state poles (natural frequencies) are located directly on the $j\omega$ axis of the s plane, as shown in Fig. 3(b), where according to (6) and (7), $H(j\omega_1) = \infty$.

Equations (7) determine the steady-state oscillation conditions and are called the **Barkhausen criterion**.

D. Initial Pole Location

We now have to decide on the pole location in the initial state. Let us first suppose that there should be no difference between the **initial and final pole locations**; in the both cases the poles can be located directly on the $j\omega$ axis.

Magnitude of the natural response (4) in this case [see Fig. 3(b)] is a function of the initial state. It is an *undesirable*

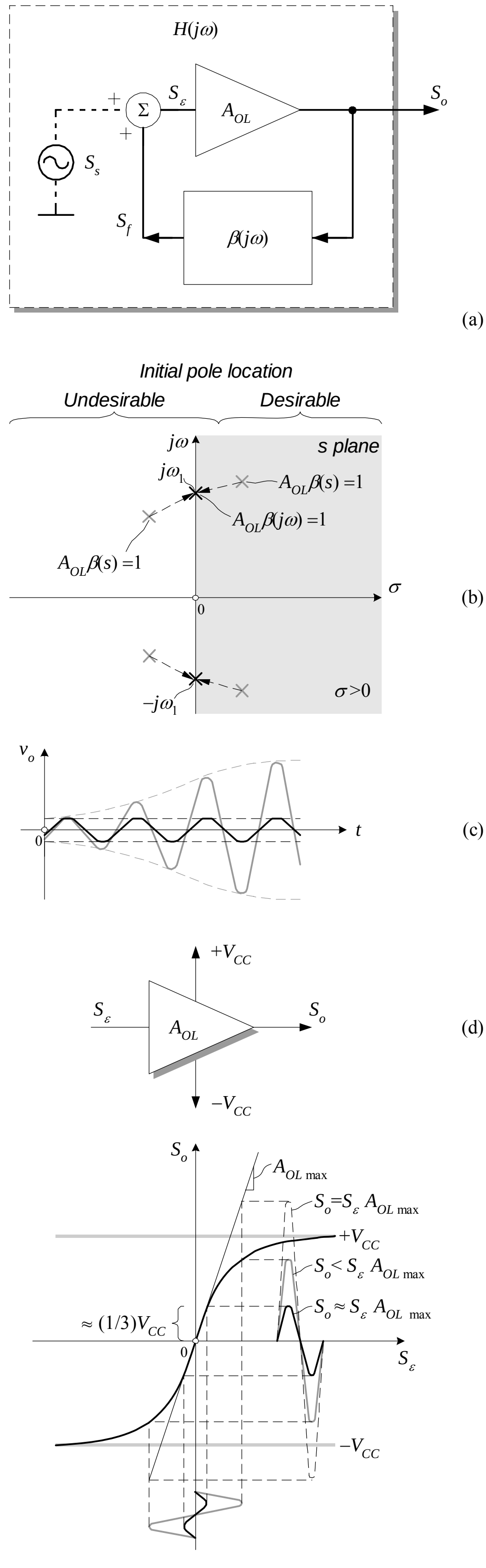

Fig. 3. Positive-feedback oscillator (a), locus of its natural frequencies as a function of the lop gain $A_{OL}\beta(j\omega)$ (b), its natural time response (c), and nonlinear properties of the amplifier A_{OL} (d).

situation for the three following reasons. First, we do not intend to supply the circuit with additional means to control the initial state. Second, any circuit is subject to disturbances and noise. In the case of an oscillator, the uncontrolled initial state, disturbances, and noise will make its steady state unpredictable. Third, it is impossible to provide the initial pole locations exactly on the $j\omega$ axis because of the non-zero tolerances of the circuit components.

Since it is undesirable to locate the oscillator's initial-state poles directly on the $j\omega$ axis, they should be located in the right half of the s plane at a distance from the $j\omega$ axis. This distance [see Fig. 3(b)] should be large enough to ensure the desirable pole location for any deviations of the circuit element values within their tolerances and temperature ranges.

The oscillator in this case will generate an increasing sinewave. When the sinewave magnitude will become sufficiently large, a **negative feedback** should stabilize it by shifting the poles towards the $j\omega$ axis, as shown in Figs. 3(b) and (c).

It is undesirable to locate the poles in the left half of the s plane because, in this case, the oscillator steady state will be zero.

To summarize the above: we want the oscillator to be **unstable** in the initial state and **conditionally stable** in the steady state. We never want it to be stable.

The simplest—but not always the best (see the concluding part of this Section)—way to determine the initial poles location in the right half of the s plane and to provide their movement to the $j\omega$ axis, when the oscillations approach their steady-state magnitude, is to use the nonlinear properties of the amplifier A_{OL}, as shown in Fig. 3(d).

In the initial state, the small-signal value of A_{OL} is simply set to be greater than it is needed to satisfy the Barkhausen criterion (7), $A_{OL} > A_{OL1}$. As a result, the initial pole location will be either to the right or to the left of the $j\omega$ axis [see Fig. 3(b)], depending on the specific behavior of the feedback transmission $\beta(s)$.

For the reasons discussed above, $\beta(s)$ for which the initial poles are in the left half of the s plane should not be employed in our design. Below (see Section III), we will show how one can easily decide whether $A_{OL} > A_{OL1}$ shifts poles to the right or to the left of the $j\omega$ axis.

E. Negative Feedback

Meanwhile, let us suppose that $A_{OL} > A_{OL1}$ shifts the poles as desired, to the right of the $j\omega$ axis. This forces unstable operation of the oscillator. It starts from an initial state and builds up its oscillations, as shown in Fig. 3(c). When the oscillations became large enough and reach nonlinear part of the amplifier transfer characteristic A_{OL}=S_o/S_ε in Fig. 3(d), the amplifier gain A_{OL} decreases (negative feedback) and approaches A_{OL1}. Accordingly, the poles move towards their steady-state location on the $j\omega$ axis, as shown in Fig. 3(c). The steady state is reached when the amplifier gain at

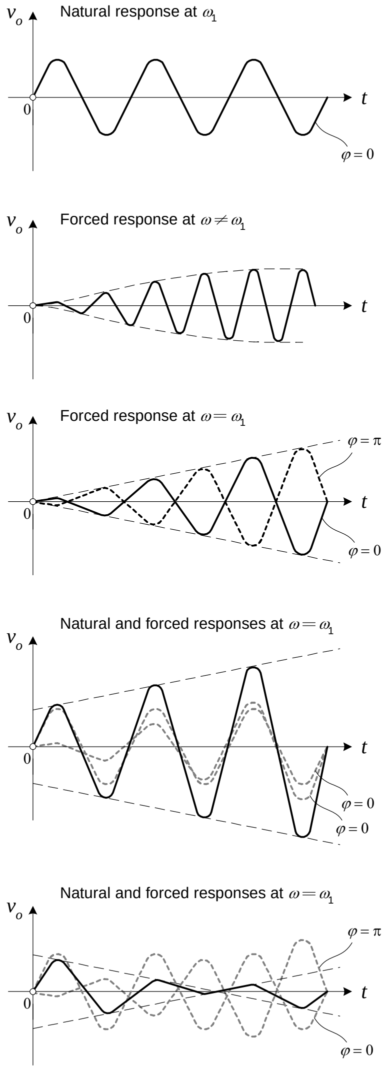

Fig. 4. The effect of disturbances and noise on the oscillator output.

frequency ω_1 equals A_{OL1}, and the Barkhausen criterion (7) is satisfied.

F. The effect of Disturbances and Noise

A disturbance and noise affecting the circuit can be translated to the input of the oscillator and represented by an equivalent source S_{in}, as shown in Fig. 3(a).

The source S_{in} will contribute to the oscillator output: it will either decrease or increase the oscillation caused by the initial state (the natural response).

Recall that a circuit having a single couple of *purely* imaginary poles is conditionally stable. (A circuit having more than a couple of equal imaginary poles has linearly increasing natural response and, hence, is unstable.) A conditionally stable circuit has a bounded steady state only if it is not excited at the frequency of the poles (see Fig. 4). If the circuit is excited at the frequency of the poles then its forced response linearly increases with time.

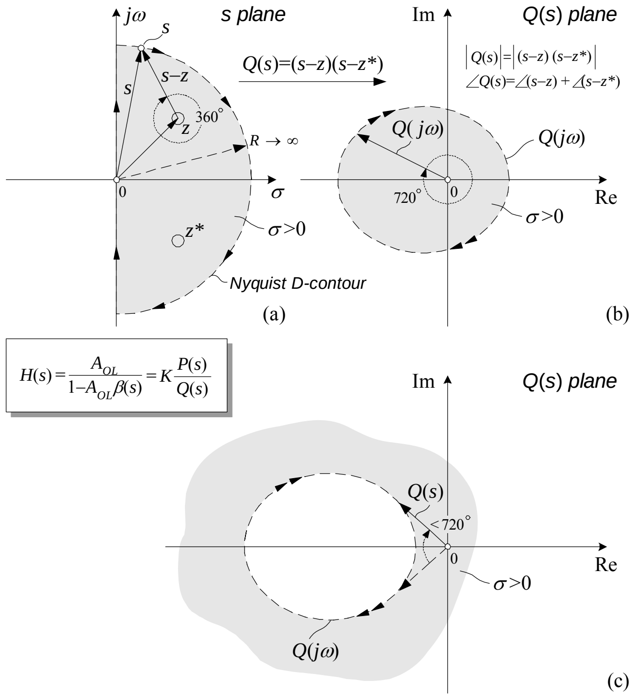

Fig. 5. Mapping of the right half of the *s* plane: the Principle of the Argument.

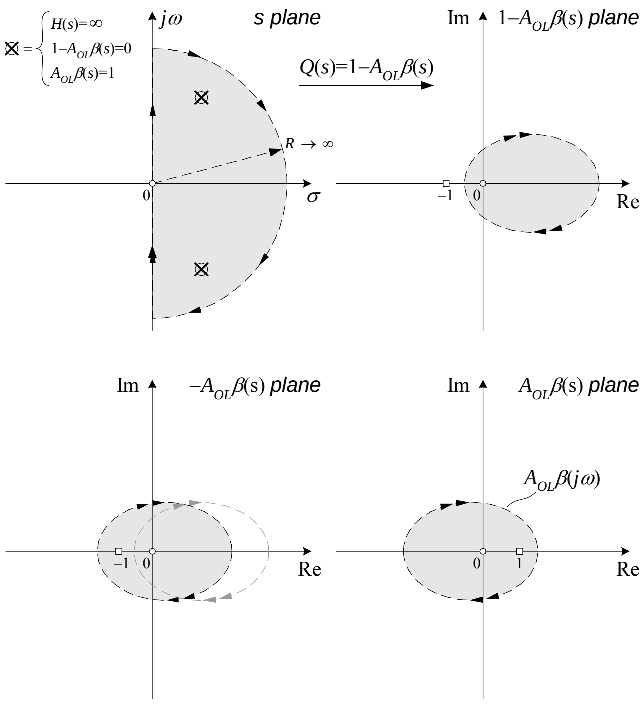

Fig. 6. The Nyquist stability criterion.

Let us suppose now that the spectrum of the disturbance and/or noise is continuous and always includes a sinusoidal excitation at the frequency of the oscillator poles. This excitation will contribute a linearly increasing part (forced response) to the oscillator natural response as shown in Fig. 4. Depending on the phase difference between the natural and forced responses, the oscillator output will either increase immediately after adding the external excitation or first decrease and after that increase (see Fig. 4).

The decreased oscillations will increase the amplifier gain A_{OL} above its steady state value A_{OL1}, and the increased gain will shift the oscillator poles in the right half of the *s* plane. The oscillator natural response will increase, compensate for the disturbance or noise, and the poles will move back towards the $j\omega$ axis (recall Fig. 3).

The increased oscillations will decrease the amplifier gain A_{OL} below its steady state value A_{OL1}, and the decreased gain will shift the oscillator poles in the left half of the *s* plane. The oscillator natural response will decay, and the oscillator steady state will be defined by its steady state *response* to the input signal S_{in} in accordance with the network function (6).

Since in the new steady state, caused by a disturbance or noise, the loop gain $A_{OL}\,\beta(j\omega_1)$ will be smaller than unity, the network function (6) will be finite, and so will be the oscillator output. The specific value of the amplifier gain A_{OL} in the new steady state and thus the specific value of the network function and the oscillator output depend on the magnitude of the disturbance or noise and on the nonlinear properties of the amplifier.

Note that under influence of a disturbances or noise, the oscillator operates as a frequency-selective amplifier with the total gain described by the network function (6). At the frequency of oscillation, the total gain is at maximum and, at the other frequencies, it decreases in accordance with the frequency behavior of the feedback transmission $\beta(j\omega)$. The disadvantage of the use of the nonlinear properties of the

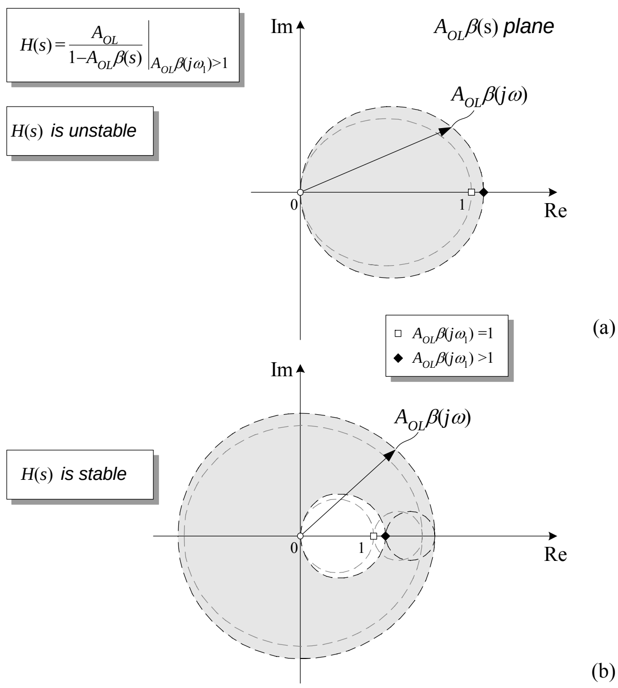

Fig. 6. Examples for an unstable (a) and stable (b) networks. Note that in the both cases the loop gain $A_{OL}\beta(j\omega_1)$ can be greater than unity.

amplifier is that its output signal becomes distorted.

To avoid distortions, it is better to control with the help of a negative feedback the value of $\beta(j\omega_1)$ in such a way that the amplifier output is always within one third of its full scale [see Fig. 3(d)]. This can be done, for example, by using in the feedback network photo- or thermo-resistors, which values depend on *average*, not instant, magnitude of oscillations.

III. THE NYQUIST STABILITY CRITERION

We shall finally illustrate how one can easily decide whether $A_{OL} > A_{OL1}$ shifts the oscillator poles from their steady-state position on the $j\omega$ axis to the right or to the left of this axis.

Let us first assume that the denominator in (6) has only two zeros:

$$Q(s) = 1 - A_{OL}\beta(s) = (s - z)(s - z^*),$$

$$|Q(s)| = |(s - z)(s - z^*)| \qquad . \qquad (8)$$

$$\varphi_{Q(s)} = \varphi_{(s-z)} + \varphi_{(s-z^*)}$$

Let us also assume that the two zeros of $Q(s)$ lie in the right half of the s plane, as shown in Fig. 5. We consider as well that for a passive $\beta(j\omega)$, $Q(s)$ has no poles in the right half of the s plane.

If we now encircle the *entire* right half of the s plane with a closed contour (so called Nyquist *D*-contour), as shown in Fig. 5(a), another closed contour, $Q(j\omega)$, is created in the complex $Q(s)$ plane around its origin, $Q(s)$=(0, j0), as shown in Fig. 5(b).

The function $Q(s)$ is said to map the right half of the s plane *inside* the contour $Q(j\omega)$ in the $Q(s)$ plane. [Note also that contour $Q(j\omega)$ in Fig. 5(b) encircles the origin of the $Q(s)$ plane in the same direction as the right half of the s plane is encircled in Fig. 5(a).]

The above result is known as the **Principle of the Argument**. This principle can be easily understood if one considers an opposite case. Consider, for example, that $Q(j\omega)$ does *not* encircle the origin of the $Q(s)$ plane, as sown in Fig. 5(c). In this case, the vector $Q(j\omega)$ will not rotate by 720^o for $-\infty < j\varpi < \infty$ as Fig. 5(a) and (8) dictate.

We are now ready to formulate the **Nyquist stability criterion** for a positive-feedback oscillator combined of a *stable* feedback network $\beta(j\omega)$ and a *stable* amplifier A_{OL}: such an oscillator is *unstable* if its loop gain $A_{OL}\beta(j\omega)$ *does* encircle the point (1, j0), as shown in Fig. 6.

Fig. 6 illustrates that $A_{OL}\beta(j\omega_1)$>1 can cause both unstable and stable operations of a circuit. Or in other words, $A_{OL}\beta(j\omega_1)$>1 can shift the circuit's poles either to the right or to the left of the $j\omega$ axis, depending on the specific behavior of the feedback transmission $\beta(j\omega)$.

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