1. **MAC with common information.** Consider a DM-MAC $P_{Y|X_1,X_2}$ with three independent uniformly distributed messages $W_0 \in [1, \ldots, 2^{nR_0}]$, $W_1 \in [1, \ldots, 2^{nR_1}]$, and $W_2 \in [1, \ldots, 2^{nR_2}]$. The first encoder maps each pair $(w_0, w_1)$ into a codeword $x^n_1(w_0, w_1)$ and the second maps each pair $(w_0, w_2)$ into a codeword $x^n_2(w_0, w_2)$. The decoder upon receiving $y^n$, finds an estimate $(\hat{w}_0, \hat{w}_1, \hat{w}_2)$ of the messages sent. The probability of decoding error is:

$$P(n)_e = \Pr \left( (\hat{W}_0, \hat{W}_1, \hat{W}_2) \neq (W_0, W_1, W_2) \right).$$ (1)

Show that the capacity region for this channel is given by the set of rate triples $(R_0, R_1, R_2)$ such that

$$R_1 \leq I(X_1; Y|X_2, U),$$
$$R_2 \leq I(X_2; Y|X_1, U),$$
$$R_1 + R_2 \leq I(X_1, X_2; Y|U),$$
$$R_0 + R_1 + R_2 \leq I(X_1, X_2; Y),$$ (2)

for some $p(u)p(x_1|u)p(x_2|u)$. You need to prove achievability and converse. (Hint: In proving the converse you may use the identification $U_i = W_0$.)

2. **Strong $\epsilon$-typicality.** Achievability proofs involving covering, e.g., for the rate distortion theorem, require that we find a good lower bound on the probability that one specific typical sequence $x^n$ is jointly typical with a randomly drawn sequence $Y^n$. Using strong typicality, the desired lower bound can be established. Let $(X_i, Y_i)$ be drawn i.i.d. $\sim P(x, y)$ and assume that the cardinalities $\mathcal{X}, \mathcal{Y}$ are finite. Let the marginals of $X$ and $Y$ be $P(x)$ and $P(y)$, respectively (you may use ideas and results from methods of types to solve the exercise). We use in this exercise a specific notation $\delta(\epsilon)$ that implies that $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. 

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**Homework Set #5**

Question 3 may increase your final grade by 10 points.
(a) Show that if \( x^n \in T_e^{(n)}(X) \), then
\[
p(x^n) \doteq 2^{n(H(X) \pm \delta(\epsilon))}.
\] (3)

(b) Show that \( \Pr(X^n \in T_e^{(n)}(X)) \to 1 \), as \( n \to \infty \).

(c) Show that
\[
|T_e^{(n)}(X)| \doteq 2^{n(H(X) \pm \delta(\epsilon))}.
\] (4)

(d) Let \( x^n \in T_e^{(n)}(X) \), and let \( T_e^{(n)}(Y|x^n) \) be the set of \( y^n \) sequences such that \( (x^n, y^n) \in T_e^{(n)}(X, Y) \). Show that
\[
|T_e^{(n)}(Y|x^n)| \doteq 2^{n(H(Y|X) \pm \delta(\epsilon))}.
\] (5)

(e) Let \( x^n \in T_e^{(n)}(X) \), and \( Y^n \) be drawn independently of \( x^n \) i.i.d. \( \sim P(y) \). Show that
\[
\Pr(x^n, Y^n \in T_e^{(n)}(X, Y)) \doteq 2^{n(I(X,Y) \pm \delta(\epsilon))}.
\] (6)

(Note that in (d) and (e) the bounds do not depend on \( x^n \).)

3. (10 points bonus to the final grade for the first two students who solve the following question)

(5 points.) **MAC with causal state information at the encoder**
Find the capacity region of a MAC \( P_{Y|X_1, X_2, S} \), where the state information \( S \) is known *causally* to both encoders but not to the decoder, as depicted in Fig. 1

( 5 points.) **MAC with non causal state information at the encoder**
Find the capacity region of a MAC \( P_{Y|X_1, X_2, S} \), where the state information \( S \) is known *noncausally* to both encoders but not to both decoder, as depicted in Fig. 1
Figure 1: Mac with state information. Find the capacity region of the MAC with state information at the encoder. The state is known causally or non-causally to the encoders.