Three-valued Karnaugh Maps

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ABSTRACT

A method is presented for the disjunctive simplification of 3-valued logic functions using 3-valued Karnaugh maps. The number of variables which can be depicted is at most four. While a particular set of primitive operations is used with the simplification method, the method is not restricted to any particular operation set, provided only that the operations of disjunction and conjunction are included.

SECTION I. BACKGROUND

The 2-valued K-map method [5] provides a simple straightforward method for minimizing Boolean functions. This method may be regarded as a pictorial improvement over the method of truth tables or Venn diagrams.

K-maps for 2-valued functions are easy to visualize, as long as the number of variables is small. Maps of more than seven variables can not be visualized easily.

The algebra for n-valued logic is called Post algebra of order n [3,8,9]. Boolean algebras are Post algebras of order 2. Some basic operations in Post algebras of order 3 are given in the table below.

X	e_0	e_1	e_2	$C_0(X)$	$C_1(X)$	$C_2(X)$	N(X)	X'
0	0	1	2	2	0	0	2	1
1	0	1	2	0	2	0	1	2
2	0	1	2	0	0	2	0	0

The prime operation X' comes from Post in 1921 [8]. The strong negation operation N(X) comes from Cukasiewicz in 1920 [6]. The operations e_i and C_i come from Epstein in 1960 [3]. The approach of this paper uses all of these columns except for the last column X'.

In Post algebras of order n, $a \vee b$ is defined as the least upper bound of a, b and $a \wedge b = ab$ is defined as the greatest lower bound of a, b.

We refer to two major discussions on multiple-valued K-maps. The first was by Epstein in 1959 and 1960 [2,3]. The second was by Givone and Snelsire in 1968 [4]. These are discussed below.

Epstein in 1959 [2] gave n-valued K-maps for $n \geq 3$. A 3-valued K-map for a 2 variable function is shown below.

	$C_0(Y)$	$C_1(Y)$	$C_2(Y)$
$C_0(X)$	0	0	0
$C_1(X)$	0	0	2
$C_2(X)$	0	1	0

This function takes the value 0 at 7 cells, the value 1 at the cell determined by $C_2(X)C_1(Y)$, and the value 2 at the cell determined by $C_1(X)C_2(Y)$. The function is thus $e_1C_2(X)C_1(Y) \vee e_2C_1(X)C_2(Y) = e_1C_2(X)C_1(Y) \vee C_1(X)C_2(Y)$. Here map entries i are represented within the function as e_i. Recall that for n = 3, $e_0 = 0$ is FALSE and $e_2 = 2$ is TRUE.

Epstein in 1959 and 1960 [2,3] allows single variables such as X, Y in the simplification, but doesn't envision the appearance of strong negation, such as N(X), N(Y). For example, Epstein allows $XC_1(Y)$ as a simplified answer, but not XN(Y). No details are given by Epstein for obtaining such simplifications.

A method of map minimization for multiple-valued logic was given by Givone and Snelsire in 1968 [4]. They used $X^{i,j}$ standing for $X^{i,j} = C_i(X) \vee C_{i+1}(X) \vee \ldots \vee C_3(X)$.

The Givone-Snelsire approach allows neither single variables such as X, Y nor strong negation, such as N(X), N(Y), in the simplification. For example $(X^{1,1})(Y^{1,2}) = C_1(X)(C_1(Y) \vee C_2(Y))$ is allowed as a simplified answer, but neither XY nor

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$N(X)N(Y)$ is allowed. It should be mentioned that there are many tabular or cubical methods in the literature which share this kind of deficiency (see for example, various simplification methods in [10,11,12]). It is shown below that there is a powerful geometric reason for allowing strong negation, $N(X)$, in K-maps.

It is the intention of this paper to present a method to minimize K-maps of 3-valued logic functions. The 3-valued, 2 variable case is treated and it is shown how this may be extended to 3-valued functions of 3 or 4 variables.

SECTION II. INTRODUCTION

The minimization of two-valued functions may be accomplished by two-valued K-maps. For a small number of variables, a function to be minimized may be mapped by expanding the expression into a full disjunctive normal form and marking a "one" on the map for all cells corresponding to fundaments of the function. The object is to obtain a minimized disjunctive form, in which the disjunction consists of a minimum number of conjunctive-i-players, and further the total number of terms is a minimum. A conjunctive-i-player is a conjunction of i terms. For example, $WX\bar{Y} \vee WXY \vee \bar{W}XY$ simplifies to $WX \vee XY$. There are two conjunctive-2-players in this answer and the total number of terms is 4.

In the K-map method, every possible combination of the input variables is represented on the map by a cell. Thus, for 2-valued functions of m variables, we have 2^m cells and for 3-valued functions we have 3^m cells. For 3-valued logic systems and their functions, we use as primitive terms the following: e_i, $C_i(X_j)$, X_j, $N(X_j)$. Each

X_j takes values $0,1,2$; $j=1,2,\ldots,m$, where there are m variables. $C_i(X_j)=2$ for $X_j=i$ and $C_i(X_j)=0$ otherwise: $i=0,1,2$. e_i represents the constant i for $i=0,1,2$ and finally $N(i)=2-i$, where the symbol "-" denotes integer subtraction. In particular $e_0=0$ is FALSE and $e_2=2$ is TRUE. These operators are shown in the previous section.

An example of a 3-valued logic function using the above operations can be considered. For m=2 variables, consider the 3-valued function of 2 variables shown in the 3x3 K-map of Section I.

As explained in Section I, $F(X,Y) = e_1C_2(X)C_1(Y) \vee C_1(X)C_2(Y)$. In $F(X,Y)$, $e_1C_2(X)C_1(Y)$ is a conjunctive-3-player,

and $C_1(X)C_2(Y)$ is a conjunctive-2-player. It should also be noted that there exist certain identities in the 3-valued case. For example, $XN(X) = e_1C_1(X)$. Also note that by definition the primitive terms are the conjunctive-1-players.

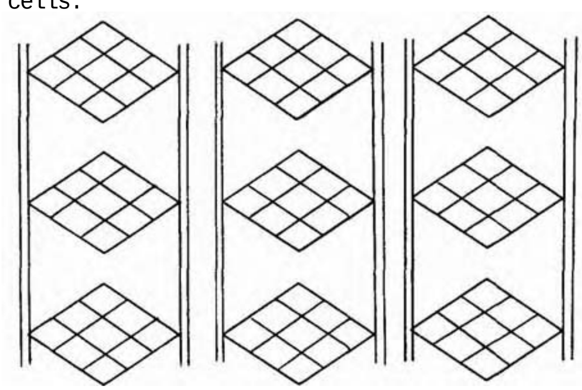
SECTION III. A METHOD TO MINIMIZE THREE-VALUED FUNCTIONS

In order to be able to systematically minimize a three-valued K-map, a simplification method is required. One must be able to apply this method to 3-valued functions of m variables and obtain the desired results, where $1 \le m < 5$. The result must be in a minimum or simplified form. The simplified disjunctive form is a disjunction of conjunctives, (i.e., a "sum of products" using the usual operations of MAX; MIN). Specifically, the number of conjunctives must be a minimum, and for this minimum number of conjunctives, the total numbers of terms must be a minimum.

In 1953 Karnaugh's method for 2-valued K-maps was successfully applied to several functions. Later work showed how to obtain a minimum expression without redundant conjunctives (see for example [7],pp.120-121,prob.4.19]).

Karnaugh's method was described for 2-valued functions. A method for 3-valued functions of m variables is developed below. This method removes redundant conjunctives whenever they occur. Before the method is discussed, we introduce some terminology.

First, the 3-valued K-map of m variables is a hypercube of 3^m cells. An example is shown below for the 3-valued, 4 variable case. In this example there are 3 =81 cells.



The number of variables is limited to 4 for obvious visual reasons.

Conjunctive-i-players are defined to be conjunctions of .terms taken i at a time. Conjunctive-1-players coincide with the primitive terms.

Consider a 3-valued function of m variables on a 3-valued K-map of 3^m cells. Denote the

3^m entries on this map by the m dimensional array M. Consider the placement of a conjunctive-i-player on such a map. For this placement, denote the corresponding entries of the player by the m dimensional array P.

For a conjunctive-i-player to be placed on a K-map, it is required that $P \leq M$ at each cell. If at one nonzero cell the value of P at that cell is equal to the value of M at that cell, the player is said to touch the function at that cell. Whenever a conjunctive-i-player is placed on a K-map, an asterisk is placed at each touched cell which does not already have an asterisk. All nonzero cells on the map which are not asterisked are called new cells. For fixed i, let a conjunctive-i-player be placed on a K-map such that the number of new cells which are touched is a maximum over all possible placements of conjunctive-i-players on the map: such a conjunctive-i-player is said to maximally touch the function.

Finally, suppose a set of players has been placed upon a K-map, touching a certain number of cells. If one of these players can be removed such that the remaining players still touch the same number of cells, then the removed player is called a redundant player or conjunctive. The method is below.

Simplification Method

Part I. For a given switching function on the K-map, initially all cells of the K-map are untouched.

(i) Using conjunctive-1-players, make all possible placements on the K-map using maximally touching players. Repeat until done. For each placement asterisk the touched cells and continue.

(ii) Using conjunctive-2-players, make all possible placements on the K-map using maximally touching players. Repeat until done. For each placement asterisk the touched cells and continue.

(iii) Continue the above steps for conjunctive-3-players, conjunctive-4-players1 etc. until all nonzero cells are touched.

Part II. Part I generates various sets of conjunctive players. Each set of conjunctive players touches all the nonzero cells of the function. However, these sets need not be distinct.

(i) Reduce these sets to distinct sets. Note that the disjunction of players in any one set yields an expression which is equal to the given switching function.

(ii) Remove redundant players in all possible ways.

(iii) Find the simplified disjunctive forms from the expressions of Part II(ii).

(iv) If there are cost preferences, select those expressions which have preferred cost. For example, e_1 might be preferred over x (since the former is internally available and does not require an input) and x might be preferred over $C_1(X)$ (since the former is an input and does not require a unary operation).

SECTION IV. 3-VALUED FUNCTIONS OF ONE VARIABLE

The method described in the previous section can be applied to the 27 three-valued functions of one variable. Since there are only 27 such functions, they can all be treated easily. Simplified disjunctive forms for all 27 three-valued functions of one variable may be found in [1]. The above method gives all of these answers exactly. In applying the method, alternative solutions and redundant players arise. Since all of the answers for this case are in [1], we rely on the remaining sections to provide examples of our method.

SECTION V. 3-VALUED FUNCTIONS OF TWO VARIABLES

In this section, 3-valued functions of two variables x, y are considered. A corresponding 3x3 K-map is shown below:

	$C_0(Y)$	$C_1(Y)$	$C_2(Y)$
$C_0(X)$			
$C_1(X)$			
$C_2(X)$			

The conjunctive-i-players for this case are determined. The method of Section III is then applied to minimize these 3-valued functions. Since there are 3^9 = 191683 possible functions, it is difficult to be exhaustive, as in Section IV. Below are the conjunctive-i-players for two variables X,Y (we exclude e_0 and e_2 as trivial).

CONJUNCTIVE-1-PLAYERS

(1) e_1 is obtained by placing the player shape

1	1	1
1	1	1
1	1	1

on the 3x3 K-map in its one position.

(2) X, Y, N(X), NY are obtained by placing the player shape:

0	0	0
1	1	1
2	2	2

on the 3x3 K-map in 4 different positions

(3) $C_i(X), C_j(Y)$ (i,j=0,1,2) are obtained by placing the player shape

2	2	2

on the 3x3 K-map in 6 different positions.

CONJUNCTIVE-2-PLAYERS

(4) $e_1X, e_1Y, e_1N(X), e_1N(Y)$ are obtained by placing the player shape:

0	0	0
1	1	1
1	1	1

on the 3x3 K-map in 4

different positions.

(5) $e_1C_i(X), e_1C_j(Y)$ (i,j,=0,1,2) are obtained by placing the player shape | 1 | 1 | 1 | on the 3x3 K-map in 6 different positions.

(6) XY, XN(Y), N(X)Y, N(X)N(Y) are obtained by placing the player shape:

0	0	0
0	1	1
0	1	2

on the 3x3 K-map in 4 different positions.

(7) $C_i(X)N(Y), XC_i(Y), C_i(X)Y, N(X)C_i(Y)$ are obtained by placing the player shape | 0 | 1 | 2 | on the 3x3 K-map in 12 different positions.

(8) $C_i(X)C_j(Y)$ (i,j=0,1,2) are obtained by placing the player shape | 2 | on the 3x3 K-map in 9 different positions.

CONJUNCTIVE-3-PLAYERS

(9) e XY, e_1XN(Y), e_1N(X)Y, e_1N(X)N(Y) are obtained by placing the player shape:

0	0	0
0	1	1
0	1	1

on the 3x3 K-map in 4 different positions.

(10) $e_1XC_i(Y), e_1C_i(X)Y, e_1N(X)C_i(X)N(Y)$ (i=0,1,2) are obtained by placing the player shape | 0 | 1 | 1 | on the 3x3 K-map in 12 different positions.

(11) $e_1C_i(X)C_j(Y)$ (i,j=0,1,2) are obtained by placing the player shape | 1 | on the 3x3 K-map in 9 different positions.

It should be noted that each player shape can be placed on the map in different positions. As a result, for each placement of a player on a map, a different name is found. Thus, for example, the player shape | 2 | 2 | 2 | can be placed on the 3x3 K-map in 6 different positions--yielding the 6 different conjunctive-1-players $C_0(X)$, $C_1(X)$, $C_2(X)$, $C_0(Y)$, $C_1(Y)$, $C_2(Y)$.

Conjunctive-i-players consist of conjunctions of i terms. In the case of this section, there is no conjunction of 4 or more terms because any such conjunction must reduce to a conjunction of 3 or less terms. For example, e XYN(X) is reducible to $e_1C_1(X)Y$. The example below demonstrates how the method is applied.

Example

The 3x3 K-map for the function E(X,Y) is given below:

	$C_0(Y)$	$C_1(Y)$	$C_2(Y)$
$C_0(X)$	2	1	0
$C_1(X)$	0	1	2
$C_2(X)$	2	1	0

I(i) No conjunctive-1-player touches the above function.

I(ii) The con'unctive-2-player shape | 1 | 1 | 1 | maximally touches this K-map at 3 cells, as shown below.

2	1*	0
0	1*	2
2	1*	0

The name for this placement is $e_1C_1(Y)$.

The three remaining new cells can be treated in any order. There are two possibilities for each of these new cells, namely the player shape | 0 | 1 | 2 | or the player shape | 2 |. The names of these placements are shown in Part II below. This completes Part I, as all nonzero cells are now touched.

II(i) The 8 distinct sets are: $\{e_1C_1(Y), C_0(X)N(Y), C_1(X)Y, C_2(X)N(Y)\}$, $\{e_1C_1(Y), C_0(X)N(Y), C_1(X)Y, C_2(X)C_0(Y)\}$, $\{e_1C_1(Y), C_0(X)N(Y), C_1(X)C_2(Y), C_2(X)N(Y)\}$, $\{e_1C_1(Y), C_0(X)N(Y), C_1(X)C_2(Y), C_2(X)C_0(Y)\}$, $\{e_1C_1(Y), C_0(X)C_0(Y), C_1(X)Y, C_2(X)N(Y)\}$, $\{e_1C_1(Y), C_0(X)C_0(Y), C_1(X)Y, C_2(X)C_0(Y)\}$, $\{e_1C_1(Y), C_0(X)C_0(Y), C_1(X)C_2(Y), C_2(X)N(Y)\}$, $\{e_1C_1(Y), C_0(X)C_0(Y), C_1(X)C_2(Y), C_2(X)C_0(Y)\}$.

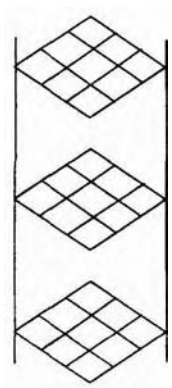
II(ii) There are no redundant players in 7 of these. However $e_1C_1(Y)$ is a redundant player in the first of these. The removal of $e_1C_1(Y)$ in the first set yields $\{C_0(X)N(Y), C_1(X)Y, C_2(X)N(Y)\}$.

II(iii) The single answer is $C_0(X)N(Y) \vee C_1(X)Y \vee C_2(X)N(Y)$.

The simplification of 3-valued functions of 1 or 2 variables by 3-valued K-maps can be extended to more than 2 variables. This is discussed in Section VI below.

SECTION VI. MINIMIZATION OF 3-VALUED FUNCTION OF THREE VARIABLES

In this section the number of variables is extended from 2 variables to 3 variables. The 3x3x3 K-map for this situation is shown below:

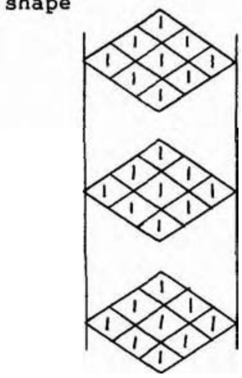


Here the top plane is $C_0(Z)$, the middle plane is $C_1(Z)$, and the bottom plane is $C_2(Z)$. The vertical planes from left-rear to right-front are $C_0(Y)$, $C_1(Y)$, and $C_2(Y)$. The vertical planes from right-rear to left-front are $C_0(X)$, $C_1(X)$, and $C_2(X)$.

The **conjunctive-i-players** for the 3-valued,, 3-variable case are below (we again exclude e_0 and e_2 as trivial).

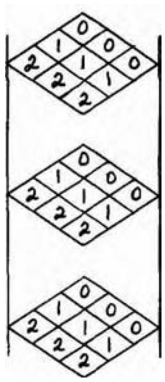
CONJUNCTIVE-1-PLAYERS

① e_1 is obtained by placing the player shape



on the **3x3x3** K-map in its one position.

② X,Y,Z,N(X),N(Y),N(Z) are obtained by placing the player shape



on the **3x3x3** K-map in 6 different positions.

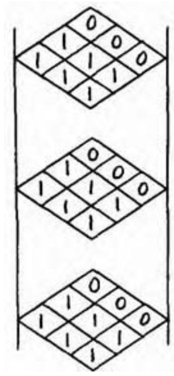
③ $C_i(X), C_j(Y), C_k(Z)$ **(i,j,k=0,1,2)** are obtained by placing the player shape

2	2	2
2	2	2
2	2	2

on the **3x3x3** K-map in 9 different positions.

CONJUNCTIVE-%PLAYERS

④ $e_1X, e_1Y, e_1Z, e_1N(X), e_1N(Y), e_1N(Z)$ are **obtained by placing** the player shape



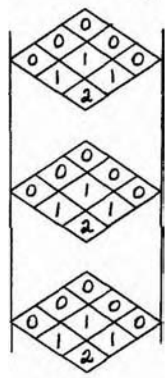
on the 3x3x3 K-map in 6 different positions.

⑤ $e_1C_i(X), e_1C_j(Y), e_1C_k(Z)$ (i,j,k=0,1,2) are obtained by placing the player shape

1	1	1
1	1	1
1	1	1

on the **3x3x3** K-map in 9 different positions.

⑥ XY, **XZ, YZ, XN(Y), XN(Z), N(X)Y, N(X)Z, YN(Z), N(Y)Z, N(X)N(Y), N(X)N(Z), N(Y)N(Z)** are obtained by placing the player shape



on the **3x3x3** K-map in 12 different positions.

⑦ $XC_i(Y)$, $XC_i(Z)$, $C_i(X)Y$, $YC_i(Z)$, $C_i(X)Z$, $C_i(Y)Z$, $C_i(X)N(Y)$, $C_i(X)N(Z)$, $N(X)C_i(Y)$, $C_i(Y)N(Z)$, $N(X)C_i(Z)$, $N(Y)C_i(Z)$ (i=0,1,2) are obtained by placing the player shape

0	1	2
0	1	2
0	1	2

on the **3x3x3** K-map in 36 different positions.

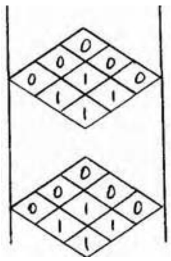
⑧ $C_i(X)C_j(Y)$, $C_i(X)C_j(Z)$, $C_i(Y)C_j(Z)$ **(i,j=0,1,2)** are obtained by placing the player shape

2	2	2

on the **3x3x3** K-map in 27 different positions.

⑨ $e_1XY, e_1XZ, e_1YZ, e_1XN(Y), e_1XN(Z), e_1N(X)Y, e_1YN(Z), e_1N(X)Z, e_1N(Y)Z, e_1N(X)N(Y), e_1N(X)N(Z), e_1N(Y)N(Z)$ are obtained by placing the player shape

on the **3x3x3** K-map in 12 different positions.



⑩ $e_1XC_i(Y)$, $e_1XC_i(Z)$, $e_1C_i(X)Y$, $e_1YC_i(Z)$, $e_1C_i(X)Z$, $e_1C_i(Y)Z$, $e_1C_i(X)N(Y)$, $e_1C_i(X)N(Z)$, $e_1C_i(Y)N(Z)$, $e_1N(X)C_i(Z)$, $e_1N(Y)C_i(Z)$, $e_1N(X)C_i(Y)$ (i=0,1,2) are obtained by

lacin the player shape

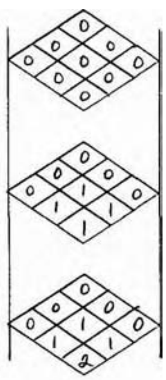
placing th

0	1	1
0	1	1
0	1	1

on the **3x3x3** K-map in 36 different positions.

⑪ $e_1C_i(X)C_j(Y)$, $e_1C_i(X)C_j(Z)$, $e_1C_i(Y)C_j(Z)$ (i,j=0,1,2) (i,j=0,1,2) are obtained by placing the player shape | 1 | 1 | 1 | on the **3x3x3** K-map in 27 different positions.

⑫ **XYZ, XYN(Z), XN(Y)N(Z), XN(Y)Z**, N(X)YZ, N(X)YN(Z), N(X)N(Y)Z, N(X)N(Y)N(Z) are obtained by placing the player shape



on the **3x3x3** K-map in **8** different positions.

⑬ $XYC_i(Z)$, $XC_i(Y)Z$, $C_i(X)YZ$, $XN(Y)C_i(Z)$, $XC_i(Y)N(Z)$, $N(X)YC_i(Z)$, $C_i(X)YN(Z)$, $N(X)C_i(Y)Z$, $C_i(X)N(Y)Z$, $N(X)N(Y)C_i(Z)$, $N(X)C_i(Y)N(Z)$, $C_i(X)N(Y)N(Z)$ (i=0,1,2) are **obtained** by **placing** the player shape

0	1	2
0	1	1
0	0	0

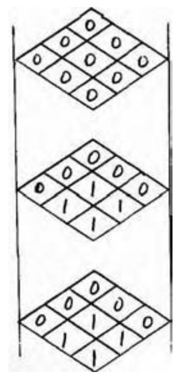
on the **3x3x3** K-map in **36** different positions.

⑭ $C_i(X)C_j(Y)Z$, $XC_i(Y)C_j(Z)$, $C_i(X)YC_j(Z)$, $N(X)C_i(Y)C_j(Z)$, $C_i(X)N(Y)C_j(Z)$, $C_i(X)C_j(Y)N(Z)$ **(i,j=0,1,2)** are obtained by **placing** the player shape | 0 | 1 | 2 | on the **3x3x3** K-map in **54** different positions.

⑮ $C_i(X)C_j(Y)C_k(Z)$ **(i,j,k=0,1,2)** are **obtained** by **placing** the player shape | 2 | on the **3x3x3** K-map in 27 different positions.

CONJUNCTIVE-4-PLAYERS

⑯ e_1XYZ, $e_1XYN(Z)$, $e_1XN(Y)Z$, $e_1XN(Y)N(Z)$, $e_1N(X)YZ$, $e_1N(X)YN(Z)$, $e_1N(X)N(Y)Z$, $e_1N(X)N(Y)N(Z)$ are obtained by placing the **player** shape



on the **3x3x3** K-map in **8** different positions.

⑰ $e_1XYC_i(Z)$, $e_1XC_i(Y)Z$, $e_1C_i(X)YZ$, $e_1XN(Y)C_i(Z)$, $e_1XC_i(Y)N(Z)$, $e_1N(X)YC_i(Z)$, $e_1C_i(X)YN(Z)$, $e_1N(X)C_i(Y)Z$, $e_1C_i(X)N(Y)Z$, $e_1N(X)N(Y)C_i(Z)$, $e_1N(X)C_i(Y)N(Z)$, $e_1C_i(X)N(Y)N(Z)$, **(i=0,1,2)** are obtained by **placing the player shape**

0	1	1
0	1	1
0	0	0

on the **3x3x3** K-map in **36** different positions.

⑱ $e_1C_i(X)C_j(Y)Z$, $e_1C_i(X)C_j(Y)N(Z)$, $e_1C_i(X)YC_j(Z)$, $e_1C_i(X)N(Y)C_j(Z)$,

$e_1 N(X) C_i(Y) C_j(Z), e_1 X C_i(Y) C_j(Z)$ $(i, j = 0, 1, 2)$

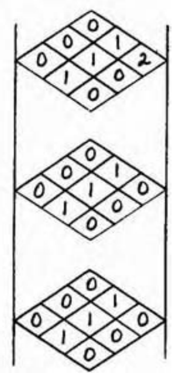
are obtained by placing the player shape |0|1|1| on the **3x3x3** K-map in 54 different positions.

(19) $e_1 C_i(X) C_j(Y) C_k(Z)$ $(i, j, k = 0, 1, 2)$ are **obtained** by[3] **placing** the player shape |1| on the **3x3x3** K-map in 27 different positions.

Recall that a conjunctive-i-player consists of a conjunction of **i** terms. In this case any conjunction of **5** or more terms is reducible to a conjunction of 4 or less terms.

The method of Section **III** is now applied. The following example demonstrates how the method is used in minimizing a 3-valued K-map for a function of 3 variables.

Example The **3x3x3** K-map for $R(X,Y,Z)$ is **given below:**

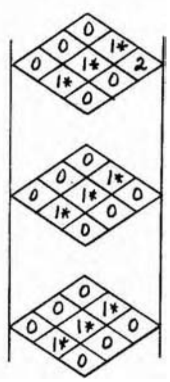


I(i) No conjunctive-1-player touches this function.

I(ii) The conjunctive-2-player

1	1	1
1	1	1
1	1	1

touches this K-map at 9 cells, as shown **below.**



The name for this placement is $e_1 C_1(Y)$. No other conjunctive-2-player **touches** this function.

I(iii) There are two different **conjunctive-**3-players which touch the remaining new cell. The player shape |0|1|1| having name $Y C_0(X) C_0(Z)$ accomplishes **this,** as does the **player** shape |2| having name $C_0(X) C_2(Y) C_0(Z)$.

This completes Part **I,** as all nonzero cells are now touched.

II(i) The 2 distinct sets are
$\{e_1 C_0(X) C_1(Y), Y C_0(X) C_0(Z)\}$ and
$\{e_1 C_0(X) C_1(Y), C_0(X) C_2(Y) C_0(Z)\}$.

II(ii) There **are** no redundant players in either of these 2 sets.

II(iii) There are two distinct answers:
$e_1 C_0(X) C_1(Y) \vee Y C_0(X) C_0(Z)$ and
$e_1 C_0(X) C_1(Y) \vee Y C_0(X) C_2(Y) C_0(Z)$.

II(iv) If, as **explained in** section **III,** Y is preferred in cost over $C_2(Y)$, then the first of the two expressions in **II(iii)** is the preferred answer. In the absence of cost **preferences,** II(iii) yields two different answers.

This example helps to demonstrate the power of our method in minimizing 3-valued functions of 3 variables. It is also possible to extend the number of variables from three to four. In order to minimize **3-**valued functions of 4 variables, the 81 cell K-map given in Section **III** is used. We refrain from giving details.

SECTION VII. SUMMARY

This paper contains several results. These results are reviewed below.

(1) A method is presented that works with any set of primitive operations, provided only that the answer is a disjunction of conjunctives **(i.e.,** a "sum-of-products" using the operations of MAX, **MIN).**

(2) This method works for the Epstein case [2,3], using besides **disjunction,** $\vee$, and conjunction, $\wedge$, constants $e_0, e_1, e_2, C_j(X_k)$, and m single variables X_k, $j = 0, 1, 2$, $k = 1, 2, \ldots, m$. In this case the operation N is withheld. For this **case,** it is only necessary to disallow rotation of players through multiples of 90 degrees, thus preventing the operation N from appearing.

(3) A method is presented which includes the operations of (2) above and also strong negation $N(X_k)$. A feature of the method with these operations is that rotations of players through multiples of 90 degrees correspond to use of the N operator. This is the first Karnaugh method that exploits such a feature.

(4) The approach of this paper limits for ease of visual comprehension the number of variables to **4** at most.

(5) The heart of the method is the use of conjunctive-i-players, which players are determined by the case at hand. For

example, there are the following distinct nontrivial conjunctive-i-player shapes for one, two, and three variables.

3-VALUED CASE	SECTION NUMBER	TOTAL NUMBER OF DISTINCT NONTRIVIAL CONJUNCTIVE-i-PLAYER SHAPES
1 VARIABLE	IV	5
2 VARIABLES	V	11
3 VARIABLES	VI	19

The complexity of determining the placement of the conjunctive-i-players depends in part on the number of different player shapes. As the number of variables increases, the number of player shapes increases and minimization becomes harder.

Finally, the simplification method in Section III works equally well for n-valued functions and n-valued K-maps, where n>3. Also, this simplification method is not restricted to the primitive operations of (2) or (3) above, provided only that the operations of disjunction and conjunction are included. It is only necessary to compute the appropriate conjunctive-i-players for the desired set of primitive operations. We plan to explore these issues in later work.

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