DIFFERENTIAL TRANSFORMS AND CIRCUIT THEORY

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SUMMARY
The function and equation transformations are presented where the transfer to images is made by means of differentiation. The application of these differential transforms for the design of linear and non-linear circuits is also discussed.

INTRODUCTION
Several problems of electrical engineering, automatization and electronics are successfully resolved using integral Laplace and Fourier transforms. These transforms are mostly used in studying systems with constant parameters. In the integral transforms the images are defined by means of function integration. In the present work the images are found using function differentiation. Therefore they are referred to as differential transforms. The mathematical formulae and rules of differential transforms allow the creation of both analytical and numerical methods for solving linear and non-linear systems.

DIRECT AND INVERSE DIFFERENTIAL TRANSFORMS
Consider the function \( x = x(t) \) of a real continuous variable \( t \). It may be represented by an absolutely convergent Taylor series within the interval \( |t-\rho| < |t| < |t+\rho| \) where \( t = t_i \) is its centre and \( \rho \) is a convergence radius. The following relations may be written:

\[
X_i(k) = M(k) \left( \frac{\partial^k x(t_i + \tau)}{\partial \tau^k} \right)_{\tau = 0} = x(t_i + \tau) = \sum_{j=0}^{k} \frac{\tau^j}{k!} \frac{X_i(k)}{M(k)}
\]

where \( M(k) \neq 0 \) is a scale function of integer argument \( k = 0, 1, 2, \ldots, \infty \), and \( \tau = t - t_i \) is the continuous argument from the point \( t = t_i \). The expression on the left side of correspondence symbol \( \approx \) is a direct transform. It allows one to find the discrete image function \( X_i(k) \) of integer argument \( k \) by means of \( x(t) = x(t_i + \tau) \). The expression on the right side of the symbol \( \approx \) is an inverse transform and it allows one to use the discrete image values \( X_i(0), X_i(1), \ldots \) to find \( x(t) \) as a Taylor power series expanded about the point \( t = t_i \).

The discrete functions \( X_i(k) \) are called T-functions and the differential transforms will be referred to briefly as T-transforms, since the functions are reconstructed in terms of discrete values of the image on the basis of Taylor’s formula.

ELEMENTARY T-FUNCTIONS
Equation (1) shows that the form of the T-function depends on the scale function \( M(k) \). Let

\[
M(k) = \frac{H^k}{k!}, \quad k = 0, 1, 2, \ldots, \infty
\]

where \( H \) is a certain positive constant with the dimension of the argument, \( t \), i.e. \( [H] = [t] \).
With such $M(k)$ it may be seen that the images of a the product of functions (as of the most usable non-linear operation) are the simplest and moreover, all the discrete values of $X_i(k)$ have the same dimension, i.e. $[X_i(0)] = [X_i(1)] = \ldots = [X_i(\infty)]$.

The transforms (1) take the form

$$X_i(k) = \frac{H^k}{k!} \left( \frac{\partial^k x(t_i + \tau)}{\partial \tau^k} \right)_{\tau=0} = x(t_i + \tau) = \sum_{k=0}^{\infty} \left( \frac{\tau}{H} \right)^k X_i(k)$$

with a scale function as given in (2).

Substitution of the elementary functions such as $1$, $t = t_i + \tau$, $t^m$, $e^{ct}$, $\sin \omega t$ and others simply defines the corresponding images.

1. The image of the function which is everywhere unity ($x(t) = 1$) has the form

$$1 = \mathcal{B}(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

The discrete function $\mathcal{B}(k)$ is called the unit or briefly 'teda'.

2. The image of the linear function ($x(t) = t$) is

$$t = t_i \mathcal{B}(k) + H \mathcal{B}(k-1) = \begin{cases} t_i, & k = 0 \\ H, & k = 1 \\ 0, & k \geq 2 \end{cases}$$

where

$$\mathcal{B}(k-1) = \begin{cases} 1, & k = 1 \\ 0, & k \neq 1 \end{cases}$$

3. The image of the $m$th power function ($x(t) = t^m$) is

$$t^m = \sum_{l=0}^{m} \binom{m}{l} t_i^{m-l} H^l \mathcal{B}(k-l),$$

where $m$ is a positive integer and

$$\binom{m}{l} = \frac{m!}{l!(m-l)!}$$

Let $t_i = 0$, then $t^m = H^{m-r} \mathcal{B}(k-m)$.

4. The image of exponential function ($x(t) = e^{ct}$) is

$$e^{ct} = \frac{(cH)^k}{k!} - e^{ct_i}$$

where $c$ is a constant.

5. The image of the sine ($x(t) = \sin \omega t$) is

$$\sin \omega t = \frac{(\omega H)^k}{k!} \sin \left( \frac{\pi k}{2} + \omega t_i \right)$$

where $\omega$ is a constant.

6. The image of the cosine ($x(t) = \cos \omega t$) is

$$\cos \omega t = \frac{(\omega H)^k}{k!} \cos \left( \frac{\pi k}{2} + \omega t_i \right)$$

The images of other functions that may be represented as absolutely convergent Taylor power series are found in a similar way.
THE CHARACTERISTICS OF DIFFERENTIAL TRANSFORMS

Some properties of differential transforms may be found directly from (3).

1. When the function \( x(t) \) is multiplied by a constant \( c \) the image should be multiplied by the same number, i.e.

\[
    cx(t) = cX_i(k)
\]

(10)

2. The algebraic sum of the functions \( x(t) = X_i(k) \) and \( y(t) = Y_i(k) \) is consistent with the same sum of their images

\[
    x(t) \pm y(t) = X_i(k) \pm Y_i(k)
\]

(11)

3. The product of the functions \( x(t) \) and \( y(t) \) is transformed into the algebraic convolution of their images

\[
    x(t)y(t) = X_i(k) \ast Y_i(k) = \sum_{l=0}^{l=k} X_i(l)Y_i(k-l)
\]

(12)

where \( \ast \) is a convolution symbol. The expression \( X_i(k) \ast Y_i(k) \) is called the \( T \)-product of the discrete functions \( X_i(k) \) and \( Y_i(k) \).

4. The quotient of the functions \( x(t) \) and \( y(t) \) is transformed into the \( T \)-function \( Z_i(k) \):

\[
    z(t) = \frac{x(t)}{y(t)} = \frac{X_i(k)}{Y_i(k)} - \sum_{l=0}^{l=k} Z_i(l)Y_i(k-l)
\]

(13)

where \( \div \) is the symbol of \( T \)-division defined by the given recurrence formula.

5. The logic-algebraic operation in the region of images

\[
    \frac{dx(t)}{dt} = \mathcal{D}X_i(k) = \frac{k+1}{H}X_i(k+1)
\]

(14)

referred to as \( T \)-differentiation, is consistent with differentiation of \( x(t) \) by the variable \( t = t_i + \tau \).

6. The logic-algebraic operation in the region of images

\[
    \int x(t) dt = \mathcal{D}^{-1}X_i(k) = \frac{H}{k}X_i(k-1) + C_i \mathcal{B}(k)
\]

(15)

is consistent with the operation of obtaining the indefinite integral of \( x(t) \) by \( dt \). Here \( C_i \) is an integration constant defined by means of additional conditions, \( \mathcal{D}^{-1} \) is the symbol of indefinite \( T \)-integration.

7. The operators \( \mathcal{D} \) and \( \mathcal{D}^{-1} \) are mutually inverse in the sense that

\[
    \mathcal{D}(\mathcal{D}^{-1}X_i(k)) = X_i(k), \quad \mathcal{D}^{-1}(\mathcal{D}X_i(k)) = X_i(k) + C_i \mathcal{B}(k)
\]

(16)

8. The value of the function \( x(t) \) at the point \( t = t_i \) is equal to the original discrete value of the image at the same point, i.e.

\[
    x(t_i) = X_i(0).
\]

(17)

9. The value of the function \( x(t) \) at the point \( t = t_i + H \) is equal to the image discrete sum at the point \( t = t_i \), i.e.

\[
    x(t_i + H) = \sum_{k=0}^{k=\infty} X_i(k).
\]

(18)

10. The value of the function \( x(t) \) at the point \( t = t_i - H \) is equal to the difference of even and odd discrete sums at the point \( t = t_i \), i.e.

\[
    x(t_i - H) = \sum_{k=0}^{k=\infty} (-1)^k X_i(k).
\]

(19)
11. The formulae
\[ \Pi X_i = X_{i+1} \tag{20} \]
\[ \Pi^{-1} X_i = X_{i-1} \tag{21} \]
are the generalizations of the two preceding formulae. Here \( X_i \) is a vector; its components being the discrete image values
\[ X_i = \begin{bmatrix} X_i(0) \\ X_i(1) \\ X_i(2) \end{bmatrix} \tag{22} \]
and \( \Pi, \Pi^{-1} \) are triangular matrices
\[ \Pi = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 1 \\ 1 & 4 & 10 & 1 \end{bmatrix}, \quad \Pi^{-1} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -2 & 3 & -4 & 5 \\ 1 & -3 & 6 & -10 \\ 1 & -4 & 10 \\ 1 & -5 \end{bmatrix} \tag{23} \]
assuming \( t_{i+1} - t_i = H \) for any \( i = 0, 1, 2, \ldots \).

12. Consider the operation of function superposition. Let \( y = y(x) \) and \( x = x(t) \); the function \( x(t) \) being represented by a Taylor series in powers of \( t \) and \( y(x) \) by a Taylor series in powers of \( x \). Then
\[ y(x) = \sum_{m=0}^{\infty} \frac{(x(t) - x(t_i))^m}{m!} \left( \frac{\partial^m y}{\partial x^m} \right)_{x=x(t_i)} \tag{24} \]
Let the image of the \( m \)th power of \( z(t) = Z(k) \) be denoted by \( Z_m^m(k) \) and the image of \( y(x) \) by \( Y_i(k) = y[X_i(k)] \). As (24) is transformed into the region of images we obtain
\[ Y_i(k) = y(X_i(k)) = \sum_{m=0}^{\infty} \frac{(X_i(k) - X_i(0))^m}{m!} \left( \frac{\partial^m y}{\partial x^m} \right)_{x=X_i(0)}, \quad Y_i(0) = y(X_i(0)) \tag{25} \]
since \( x(t_i) = X_i(0) \).
Equation (25) shows that in order to obtain the image of a complex function \( y(x), x(t) \) should be replaced by \( X_i(k) \) and all usual operations by T-operations.

13. The image of \( x(t) \) shifted by a constant \( h \) is
\[ x(t \pm h) = e^{\pm h T} X_i(k) = \left( 1 \pm h T + \frac{(h T)^2}{2!} \pm \frac{(h T)^3}{3!} + \ldots \right) X_i(k) \tag{26} \]
where
\[ T^m X_i(k) = \frac{(k + 1)(k + 2) \ldots (k + m)}{H^m} X_i(k + m) \tag{27} \]
is the \( m \)th T-derivative of the function \( X_i(k) \).

14. Any T-function may be expanded in the following series over teds
\[ X(k) = \sum_{m=0}^{\infty} X(m) T(k - m), \quad m = 0, 1, 2, \ldots, \infty. \tag{28} \]

15. The formula
\[ x(ct) = e^{k T} X_i(k) \tag{29} \]
allows one to obtain the image of a function \( x(ct) \) in terms of the known image of \( x(t) \), where \( c \) is a constant.
GENERAL T-EQUATIONS OF ELECTRIC CIRCUITS

The characteristics of electric circuits with lumped values are generally defined by the system of non-linear implicit differential equations

\[ y(t, x(t), \frac{dx(t)}{dt}, \ldots, \frac{d^m x(t)}{dt^m}) = 0 \]

where \( x(t) \) is the vector of voltages, currents and of other circuit parameters, \( y \) is the operator to show what mathematical operations are to be made with \( x(t) \), its derivatives over \( t \) and with other functions of \( t \), e.g. source voltages and currents, variable resistances, variable inductance etc.

Any functions of time in circuit differential equations are assumed to be represented as absolutely convergent Taylor series in powers of \( t \), and current-voltage, flux-current, charge-voltage and other characteristics of non-linear elements may be described by similar Taylor series of current, voltage, charge, flux-linkage powers. In this case, to transfer (30) to the image region it is reasonable to change formally all functions for their images and the usual mathematical operations for corresponding T-operations (T-multiplication, T-division, T-differentiation etc.). After dividing the integration interval into \( n \)-parts and performing all the operations given above, the T-equations of the circuit are obtained

\[ y(T_i(k), X_i(k), \mathcal{D}X_i(k), \ldots \mathcal{D}^m X_i(k)) = 0 \]

where \( i = 0, 1, 2, \ldots, n \). These equations combined with initial conditions

\[ X_0(0) = x_0, X_0(1) = Hx'_0, X_0(2) = \frac{H^2}{2!}x''_0, \ldots, X_0(m-1) = \frac{H^{m-1}}{(m-1)!}x''''_0 \]

are used to estimate the transient response of the circuit. These same equations being used together with periodicity conditions \( x(t_0 + T) = x(t_0) \) taking the form

\[ X_{i+n}(k) = X_i(k) \]

in the image region for the given case, solve the problem of periodic response calculation; here \( i \) may be taken as 0 and \( n \) corresponds to the period end.

If the centre of function expansion in power series is chosen not at the origin of co-ordinates, \( (t_i \neq 0) \), then coupling equations of \( X_0 \) and \( X_i \) similar to (20) and (21) should be added to above equations.

METHOD OF LOCAL EQUATIONS

Power series are known to closely approximate analytical functions in the immediate neighbourhood of the expansion centre at \( |t - t_i| \ll \rho \), where \( \rho \) is a convergence radius.

When the interval over which the differential equations are to be solved is large enough, the power series may converge rather slowly. Then the solution may be obtained as follows.

The integration section \( (0, t_n) \) is divided into some intervals and for each point its own local T-equation is composed. Any of the local equations serve to approximate the solutions in the neighbourhood of its expansion centre. When the coupling equation is added to local equations, a system is obtained which approximates the unknown solution better than in the case of a single equation.

If the interval \( (0, t_n) \) is divided into \( n \) equal parts and the scale constant is \( H = t_n/n \), then the total system of T-equations consists of local coupling ‘forward’ equations (20) and of ‘backward’ coupling equations (21). The system of T-equations thus obtained is solved further together with initial conditions or with periodicity conditions with respect to initial discrete values \( X_i(0) \) of local images \( X_i(k) \). These values represent simultaneously the values of \( x_i \) at the same points because \( x_i = X_i(0) \).

With discrete function values \( X_i(k) \) available it is easy to represent the solution as a system of local power series.

The method of local T-equations may be extended to circuits with piecewise characteristics (current-voltage and others) of non-linear elements. Let us show this. The function which is referred to as functional
where $\alpha_{m+1} > \alpha_m$. Outside the interval $(x = \alpha_m, x = \alpha_{m+1})$ the function is zero, within the interval it is unity and at the bounds where the function is interrupted it equals $1/2$.

The piecewise function $\varphi(x)$ which is equal to the analytical function $\varphi_m(x)$ within the interval $(\alpha_m, \alpha_{m+1})$ may be written as

$$\varphi(x) = \sum_{m=0}^{m=n-1} \varphi_m(x) \gamma(x), \quad m = 0, 1, 2, \ldots, n - 1.$$  \hspace{1cm} (35)

Let $\varphi_m(x) = \phi_m(k)$. Assuming that $x = x(t)$ has image $X_i(k)$ and taking into account $x(iH) = X_i(0)$ we may write

$$\varphi(x) = \phi_i(k) = \sum_{m=0}^{m=n-1} \phi_m(k) \gamma(X_i(0)), \quad m = 0, 1, 2, \ldots, n - 1.$$  \hspace{1cm} (36)

At the conjugation points of the function $\varphi_m(x)$ with the functions $\varphi_{m-1}(x)$ and $\varphi_{m+1}(x)$ the expression (36) loses its meaning. The error appearing may, however, be suppressed by reducing the stepsize $H$ when approaching the transition point.

**EXAMPLES**

The differential transforms of electric circuit calculation are applied to some simple examples.

**Example 1**

Let the voltage and current sources represented as absolutely convergent Taylor power series that are present in the circuit be $R$, $C$, $L$. The initial values of inductance currents and capacitor voltages are assumed to be known, then T-equations of the circuit are to be composed and solved.

It is known$^1$ that the differential circuit equations may be composed as

$$\frac{dx(t)}{dt} + Ax(t) = f(t), \quad x(0) = x_0$$  \hspace{1cm} (37)

where $x(t)$ is a vector of unknown values, $A$ is a matrix of constant factors, $f(t)$ is a vector of right side.

The expansion centre of power series is chosen at the point $t = 0$. Then the T-equations of the circuit are

$$\mathcal{D}X(k) + AX(k) = F(k), \quad X(0) = x_0,$$  \hspace{1cm} (37a)

where

$$X(k) = \frac{H^k}{k!} \frac{\partial^k x(t)}{\partial t^k} \bigg|_{t=0}, \quad F(k) = \frac{H^k}{k!} \frac{\partial^k f(t)}{\partial t^k} \bigg|_{t=0}$$

The T-derivative $\mathcal{D}X(k)$ is taken and we obtain

$$\frac{k+1}{H} X(k+1) + AX(k) = F(k), \quad X(0) = x_0.$$  \hspace{1cm} (38)
This expression allows us to calculate successively the discrete values of the function of \( X(k) \). Indeed, assuming \( k = 0, 1, 2, \ldots \) we obtain

\[
X(0) = x_0, \quad X(1) = H(F(0) - AX(0)), \quad X(2) = \frac{H}{2}(F(1) - AX(1)),
\]

e tc.

The function \( x(t) \) relevant to these discrete values is obtained by means of the inverse transformation

\[
x(t) = \sum_{k=0}^{\infty} \left( \frac{t}{H} \right)^k X(k) = X(0) + \frac{t}{H} X(1) + \left( \frac{t}{H} \right)^2 X(2) + \ldots = e^{-At}x_0 + \int_0^t e^{-A(t-\eta)} f(\eta) \, d\eta \quad (39)
\]

This is a well known analytical solution of the above system of differential equations.

**Example 2**

Along with the constant parameters \( R, C, L \), as voltage and current sources, the electrical circuit has also a time-varying resistor with \( r = r(t) \), of which the image exists:

\[
R(k) = \frac{H^k}{k!} \left( \frac{\partial^k r(t)}{\partial t^k} \right)_{t=0}
\]

The T-equations of the circuit are to be composed and solved in the form of truncated power series.

Consider the \( RLC \) circuit as a one-port network and \( r(t) \) as a load on this one-port network; then the differential equations are composed.

\[
[Z(p) + r(t)]i(t) = e(t), \quad p = \frac{d}{dt} \quad (40)
\]

where \( e(t) \) is the idle voltage of two-terminal network and \( Z(p) \) is a rational fraction of the form

\[
Z(p) = \frac{a_0 + a_1 p + a_2 p^2 + \ldots + a_m p^m}{b_0 + b_1 p + b_2 p^2 + \ldots + b_n p^n}
\]

where \( a_i, b_i \) are constants.

Substituting \( p \)-operators by \( T \)-derivatives in (40) and using the notations

\[
E(k) = \frac{H^k}{k!} \left( \frac{\partial^k e(t)}{\partial t^k} \right)_{t=0}, \quad I(k) = \frac{H^k}{k!} \left( \frac{\partial^k i(t)}{\partial t^k} \right)_{t=0}
\]

we can write

\[
\left( \sum_{s=0}^{s=m} a_s T^s \right) I(k) + \left( \sum_{s=0}^{s=n} b_s T^s \right) (R(k) * I(k)) = \left( \sum_{s=0}^{s=n} b_s T^s \right) E(k) \quad (41)
\]

where

\[
T^s I(k) = \frac{(k+1)(k+2) \ldots (k+s)}{H^s} I(k+s)
\]
\[
T^s E(k) = \frac{(k+1)(k+2) \ldots (k+s)}{H^s} E(k+s)
\]
\[
R(k) * I(k) = \sum_{l=0}^{l=k} R(l) I(k-l)
\]

The T-equation (41) solves the problem of estimation of the transient and periodic responses in the above circuit with time-varying resistance \( r(t) \).
Let, for example, \( e(t) = E = \text{const} \), \( Z(p) = pL \) and
\[
r(t) = R_0(1 + \alpha \sin \omega t) = R_0 \left( \gamma(k) + \frac{\omega H}{k!} \sin \frac{\pi k}{2} \right), \quad |\lambda| < 1.
\]
Here (41) becomes
\[
L \frac{k+1}{H} I(k+1) + R_0 \sum_{l=0}^{k} \left( \gamma(l) + \frac{\omega H}{l!} \sin \frac{\pi l}{2} \right) I(k-l) = E \gamma(k),
\]
whence
\[
I(0) = i_0, \quad I(1) = \frac{H}{L} (E - R_0 I(0)), \quad I(2) = -\frac{H \lambda R_0}{l} (I(1) + \omega HI(0))
\]
With \( i_0 \) known all the remaining discrete values \( I(k) \) are evidently successively defined. When a periodic solution is sought \( I(0) \) is obtained from
\[
\sum_{k=0}^{k=\infty} I(k+1) = 0 \tag{42}
\]
which represents the periodicity condition \( i(T) = i(0) \) if \( H = T \). In both cases the solution has the form
\[
i(t) = \sum_{k=0}^{k=\infty} \left( \frac{t}{H} \right)^k I(k) = I(0) + \frac{t}{H} I(1) + \left( \frac{t}{H} \right)^2 I(2) + \ldots.
\]

Example 3

Let the synchronous generator be connected with buses of infinite power through a transmission line. The motion of the generator rotor is known to be described under some assumptions in terms of the non-linear differential equation
\[
\frac{d^2 \varphi(t)}{dt^2} + a \frac{d \varphi(t)}{dt} + b \sin \varphi(t) = p, \quad \varphi(0) = \varphi_0, \quad \left( \frac{d \varphi}{dt} \right)_{t=0} = \varphi_0', \quad 0 \leq t \leq t_n \tag{43}
\]
where \( a, b, p \) are constants and \( (0, t_n) \) is the interval within which the solution is sought.

By transferring into the image region with centres at the points
\[
t_i = iH = t_n \frac{i}{n}, \quad i = 0, 1, 2, \ldots, n-1,
\]
the system of T-equations is obtained
\[
\frac{(k+1)(k+2)}{H^2} \phi_i(k+2) + a \frac{k+1}{H} \phi_i(k+1) + b \sin \phi_i(k) = p \gamma(k), \tag{44}
\]
where the two first discrete values of the image \( \phi_i(k) \equiv \phi(iH + \tau) \) at \( i = 0 \) are known and are given by
\[
\phi_0(0) = \phi_0, \quad \phi_0(1) = H \phi_0'.
\]
The image \( \sin \phi_i(k) \) is obtained from following considerations. Note that
\[
\mathcal{D} \sin \phi_i(k) = \cos \phi_i(k) \ast \mathcal{D} \phi_i(k), \quad \mathcal{D} \cos \phi_i(k) = -\sin \phi_i(k) \ast \mathcal{D} \phi_i(k).
\]
It is easy to generate the following recurrence formulae
\[
\sin \phi_i(k) = \sum_{l=0}^{l=k-1} \left( 1 - \frac{l}{k} \right) \phi_i(k-l) \cos \phi_i(l) \tag{45}
\]
\[
\cos \phi_i(k) = -\sum_{l=0}^{l=k-1} \left( 1 - \frac{l}{k} \right) \phi_i(k-l) \sin \phi_i(l)
\]
where on the basis of (17) characteristics
\[\sin \phi_i(0) = \sin \phi_i(0),\quad \cos \phi_i(0) = \cos \phi_i(0),\quad i = 0, 1, 2, \ldots, n - 1.\]

Now assume in the above formulae \(i = 0\) and \(k = 0, 1, 2, 3, \ldots\), and the discrete values are successively obtained

\[
\begin{align*}
\phi_0(0) &= \phi_0, \\
\phi_0(1) &= H\phi_0', \\
\phi_0(2) &= \frac{H^2}{2} \left( p - \frac{a}{H} \phi_0(1) - b \sin \phi_0(0) \right), \\
\phi_0(3) &= -\frac{H^2}{6} \left( 2a \phi_0(2) + b\phi_0(1) \cos \phi_0(0) \right), \\
\phi_0(4) &= -\frac{H^2}{12} \left[ 3a H \phi_0(3) + b(\phi(2) \cos \phi_0(0) - \frac{1}{2} \phi_0^2(1) \sin \phi_0(0) \right].
\end{align*}
\]

In terms of the discrete values \(\phi_0(k)\) the power series may be found

\[
\phi(\tau) = \sum_{k=0}^{\infty} \left( \frac{\tau}{H} \right)^k \phi_0(k) = \phi_0(0) + \frac{\tau}{H} \phi_0(1) + \left( \frac{\tau}{H} \right)^2 \phi_0(2) + \ldots,
\]

which represent the solution in the neighbourhood of the point \(t_0 = 0\).

In order to obtain the power series as a solution in the neighbourhood of the point \(t_1 = H\) we do as follows. Using ‘forward’ coupling equations

\[
\begin{align*}
\phi_1(0) &= \phi_0(0) + \phi_0(1) + \phi_0(2) + \ldots, \\
\phi_1(1) &= \phi_0(1) + 2\phi_0(2) + 3\phi_0(3) + \ldots
\end{align*}
\]

the discrete values \(\phi_1(0)\) and \(\phi_1(1)\) are obtained. All remaining discrete values \(\phi_1(k)\) are found in exactly the same way, as it was with \(\phi_0(k)\) in terms of two initial discrete values \(\phi_0(0)\) and \(\phi_0(1)\).

With the help of the discrete functional values \(\phi_1(k)\) the power series is obtained

\[
\phi(H + \tau) = \sum_{k=0}^{\infty} \left( \frac{\tau}{H} \right)^k \phi_1(k) = \phi_1(0)
\]

which represents the solution in the neighborhood of the point \(t_1 = H\).

Coupling ‘backward’ equations

\[
\begin{align*}
\phi_0(0) &= \phi_1(0) - \phi_1(1) + \phi_1(2) - \phi_1(3) + \ldots, \\
\phi_0(1) &= \phi_1(1) - 2\phi_1(2) + 3\phi_1(3) - \ldots, \\
\phi_0(2) &= \phi_1(2) - 3\phi_1(3) + \ldots
\end{align*}
\]

make it possible to see how advantageously the number of discrete function values \(\phi_0(k)\) and the step size \(H\) were chosen.

The transfer to power series for solutions in the neighbourhood of the points \(t_2 = 2H, t_3 = 3H, \ldots, t_n = nH\) is analogous.

**Example 4**

It is necessary to estimate the current in the circuit in Figure 1(a) under the assumption that the diode characteristics (Figure 1(b)) are given by

\[
R_d = \begin{cases} r & \text{if } i(t) > 0 \\ R & \text{if } i(t) < 0 \end{cases}
\]

and that of voltage source is defined by the relation

\[e(t) = E \cos \omega t, t \in [t_0, t_n], t_0 = 0, \quad i(t_0) = 0.\]
Using the methods for function interruptors (34), the differential equations of the circuit with piecewise-linear characteristics of the non-linear element (46) can be written in the form of a single formula

\[
L \frac{di(t)}{dt} + \left[ r \gamma (i(t)) + R \frac{\gamma}{-\infty} (i(t)) \right] i(t) = E \cos \omega t, \quad i(0) = 0, 0 \leq t \leq t_n.
\]

While transferring into the T-area we can obtain for the interval of independent variable \( t \in [sH, (s + 1)H] \)

\[
L \frac{k+1}{H} I_s(k+1) + \left[ r \gamma (I_s(0)) + R \frac{\gamma}{-\infty} (I_s(0)) \right] I_s(k) = E \frac{(\omega H)^k}{k!} \cos \left( \frac{\pi k}{2} + s\omega H \right)
\]

where \( s = 0, 1, 2, \ldots, n-1 \) and \( I_0(0) = 0 \).

In terms of above discrete values \( I_s(k) \) the current values \( i(t) \) can be estimated at the moment \( t = (s + 1)H \) using the coupling 'forward' equation (20)

\[
i(sH + H) = I_{s+1}(0) = I_s(0) + I_s(1) + I_s(2) + \ldots
\]

The calculations in terms of coupling 'backward' equations (21)

\[
I_{s-1}(0) = I_s(0) - I_s(1) + I_s(2) - I_s(3) + \ldots
\]

\[
I_{s-1}(1) = I_s(1) - 2I_s(2) + 3I_s(3) - \ldots
\]

are checked only at \( I_s(0) > 0 \) or at \( I_s(0) < 0 \) since at \( I_s(0) = 0 \) the diode current–voltage characteristics cannot be represented by Taylor series.

The set of local power series

\[
i(sH + \tau) = \sum_{k=0}^{\infty} \left( \frac{\tau}{H} \right)^k I_s(k)
\]

defines the circuit current at any instant \( t = sH + \tau, \tau \in [0, H] \) within the interval \( (0, t_n) \).

**Example 5**

Let a periodic saw-tooth voltage be applied to a complex linear two-terminal network with \( R, L, C \) elements. The voltage is varied linearly \( e(t) = Et/T \) with period \( T \) (Figure 2). The steady state input current \( i(t) \) is to be defined.
In a circuit with constant parameters the operator $p = \frac{d}{dt}$ is substituted by the T-derivative and currents and voltages are substituted by their images to obtain T-equations from differential equations. When the transition function of the two-terminal network is $Z(p)$, the T-equation of the circuit has the form

$$Z(\mathcal{D})I_s(k) = E_s(k),$$

where $Z(\mathcal{D})$ is a rational fraction of the type

$$Z(\mathcal{D}) = \frac{A(\mathcal{D})}{B(\mathcal{D})} = \frac{\sum_{i=0}^{n} a_i \mathcal{D}^i}{\sum_{i=0}^{m} b_i \mathcal{D}^i}.$$

Let $H = T$ and an expansion centre be at the beginning of a period $(t_e = 0)$. Then

$$\left(\sum_{i=0}^{n} a_i \mathcal{D}^i\right) I(k) = \left(\sum_{i=0}^{m} b_i \mathcal{D}^i\right) E^T b(k-1).$$

When $H = T$

$$\mathcal{D}^l b(k-1) = \begin{cases} b(k)/T, & l = 1, \\ 0, & l > 0, \end{cases}$$

is taken into account, the T-equation becomes

$$(a_0 + a_1 \mathcal{D} + a_2 \mathcal{D}^2 + \ldots + a_n \mathcal{D}^n) I(k) = b_0 E^T b(k-1) + b_1 E^T b(k).$$

To solve this equation it is reasonable to use the method of normal fundamental functions. Because of the linearity of (48) it is evident that

$$I(k) = J(k) + \phi_0(k) I(0) + \phi_1(k) I(1) + \ldots + \phi_{n-1}(k) I(n-1)$$

The functions $J(k)$ and $\phi_m(k)$ are found directly from (48), so

$$J(k) = I(k) \text{ if } I(0) = I(1) = \ldots = I(n-1) = 0;$$

$$\phi_0(k) = I(k) \text{ if } I(0) = 1, I(1) = I(2) = \ldots = I(n-1) = 0; b_0 = b_1 = 0;$$

$$\phi_1(k) = I(k) \text{ if } I(0) = 0, I(1) = 1, I(2) = \ldots = I(n-1) = 0; b_0 = b_1 = 0.$$

When $J(k)$ and $\phi_m(k)$ are defined and the periodicity conditions

$$\Pi \left[ \begin{array}{c} I(0) \\ I(1) \\ I(2) \\ \vdots \end{array} \right]_{t=0}^{T} = \left( \begin{array}{c} I(0) \\ I(1) \\ I(2) \\ \vdots \end{array} \right)_{t=T}$$

are used for calculations, the discrete values $I(0), I(1), \ldots, I(n-1)$ are found to provide the periodic response. Note that the above periodicity conditions hold only for two terminal networks where input current $i(t)$ and its first $(n-1)$ derivatives over $t$ are continuous and

$$\left[ \frac{d^m i(t)}{dt^m} \right]_{t=0}^{T} = \left[ \frac{d^m i(t)}{dt^m} \right]_{t=T},$$

where $m = 0, 1, 2, \ldots, n-1$.

CONCLUSION

Even the small set of rules and formulae from the theory of differential transforms that were presented above shows that together with the well known integral transforms the differential transforms widen the
facilities for solving complex physical systems with the help of methods of mathematical simulation using computers.

The author advises use of the differential transforms for analysis and synthesis of electrical circuits and various electronic and automatic sets.

REFERENCES