AN ANALYSIS OF THE WANG ALGEBRA OF NETWORKS(1)

BY

R. J. DUFFIN

1. **Applications of Wang algebra.** An algebra with the property that

$$x + x = 0 \quad \text{and} \quad x \cdot x = 0 \tag{1}$$

for each element x of the algebra is termed a Wang algebra. This algebra gives a very interesting method of determining the basic functions associated with an electric network. (These functions are termed discriminants.)

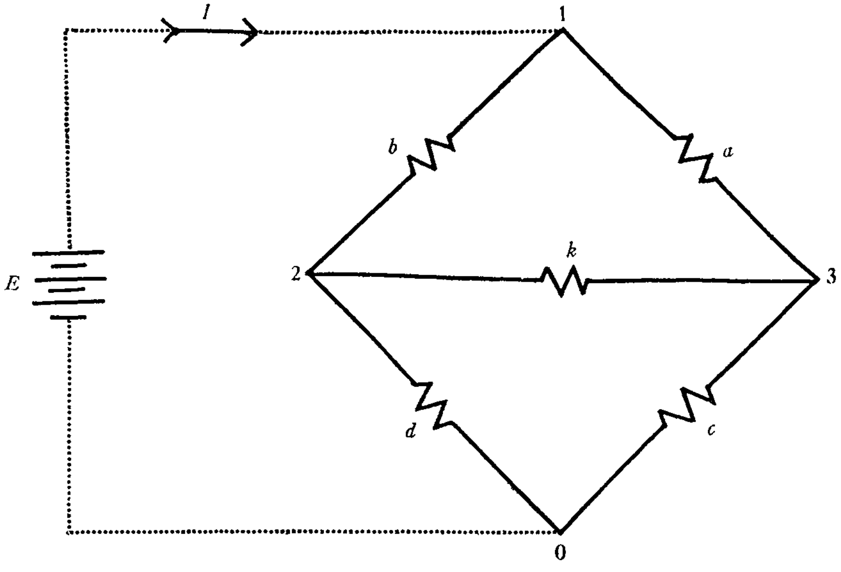

FIG. 1

The application of the Wang algebra to networks can be illustrated by the problem of the joint resistance of the Wheatstone bridge. The diagram of the Wheatstone bridge network is shown in Figure 1. The letters a, b, c, d, and k designate the five *branches* of the network. The numbers 0, 1, 2, 3 designate the four *junctions* of the network. The problem of interest is the determination of the current I flowing when a battery of potential difference E is connected

Presented to the Society, December 29, 1950 by R. J. Duffin and Raoul Bott under the title *The Wang algebra of networks*; received by the editors August 11, 1958.

(1) Part of the work on this paper was supported by the Office of Ordnance Research, U. S. Army, Contract DA-36-061-ORD-490.

between junctions 0 and 1. By Ohm's law $E=IR$ where R designates the *joint resistance* of all the branches between junctions 1 and 2.

A *star* of a network is defined as the branches meeting at a given junction. Thus the star at junction 3 consists of the branches a, c, and k. Let the branches of the network be regarded as independent generators of a Wang algebra. A *star element* of the algebra consists of the sum of the associated branches. Thus the star element at junction 3 is $a+c+k$. The element S_1 is defined as the Wang product of all star elements except those at junctions 1 and junctions 0. The element S is defined as the Wang product of all star elements except one. (It makes no difference which one is omitted.) Then the joint resistance R between junctions 0 and 1 is symbolized by

$$R = \frac{S_1}{S} \cdot \tag{2a}$$

Here S_1 and S are to be simplified by carrying out the indicated operations and making use of the Wang rules (1). After that the Wang algebra is dropped; the resulting polynomials are considered to be ordinary polynomials, and the symbols $a, b, c, \cdots$, etc., are taken to be the conductances of the corresponding branches. (In the trivial case when there are only two junctions, S_1 is to be given the value 1.)

Formula (2a) can be illustrated by the application to the network of Figure 1. First $S_1=(b+d+k)(a+c+k)$. Carrying out this product and making use of the Wang rules (1) gives

$$S_1 = ab + bc + bk + ad + cd + dk + ak + ck.$$

Next $S=(a+b)S_1$, so again making use of the Wang rules gives

$$\begin{aligned} S = {} & 0 + abc + abk + 0 + acd + adk + 0 + ack \\ & + 0 + 0 + 0 + abd + bcd + bdk + abk + bck. \end{aligned}$$

Here the terms abk cancel by (1). Substituting in (2a) gives

$$R = \frac{ab + bc + bk + ad + cd + dk + ak + ck}{abc + acd + adk + ack + abd + bcd + bdk + bck} \cdot \tag{3}$$

That this is correct can be checked by solving the Kirchhoff equations. In the present case it is convenient to do this by expressing Kirchhoff's first law (the net current entering a junction point vanishes) at the junctions 1, 2, and 3. This gives three equations in the potentials E, E_2, and E_3 of junctions 1, 2, 3 relative to junction 0.

$$\begin{aligned} I &= (a + b)E && - b\,E_2 && - a\,E_3 \\ 0 &= \quad - b\,E && + (b + d + k)E_2 && - k\,E_3 \\ 0 &= \quad - a\,E && - k\,E_2 && + (a + c + k)E_3. \end{aligned}$$

The application of Cramer's rule yields (3). In fact S is precisely the determinant of these equations. This determinant expressed as a function of the conductances of the branches is termed the *discriminant* of the network.

The Wang algebra may be used to evaluate any symmetric determinant. It is simply necessary to make a substitution to transform the determinant to network type. For example to evaluate

$$S = \begin{vmatrix} e' & b & a \\ b & d' & k \\ a & k & c' \end{vmatrix}$$

let $-e=e'+a+b$, $-d=d'+b+k$, and $-c=c'+a+k$. Then S is obtained by forming the Wang product

$$-S = (a+b+e)(b+k+d)(a+k+c).$$

Carrying out this product according to (1) gives the determinant S expressed as a function of a, b, k, c, d, e.

The rules (1) were originally proposed merely as a short-cut method and not as an algebra; see K. T. Wang [1]. Wang's method has been studied by several writers [2; 3; 4; 5; 6]. The method just given here is somewhat different from that of Wang. The difference will now be brought out.

Wang's treatment is based on meshes rather than stars. A *mesh* is a set of branches which forms a simple closed circuit. For example branches (A, B, K) form a mesh of the network of Figure 1. Here we use capital letters in place of small letters to designate the branches in order to avoid confusion with the previous method. The sum of the mesh (A, B, K) and the mesh (C, D, K) under addition mod 2 is manifestly the mesh (A, B, C, D). A set of meshes is said to be a *basis* if they are independent under addition mod 2 and if any mesh can be expressed as a sum of the meshes of the basis under addition mod 2 (see [19]). In the network shown (A, B, K) and (C, D, K) form a basis because (A, B, C, D) is the only other mesh.

Again a Wang algebra is set up with A, B, C, etc., as independent generators. A *mesh element* of the algebra consists of the sum of the branches of a mesh. Then the joint resistance between two junctions is symbolized by

$$R = \frac{M_1}{M}\cdot \tag{2b}$$

Here M is the Wang product of a basis of mesh elements and $M_1=MP$ where P is the sum of any branches forming a path between the junction 1 and junction 0. The indicated operations are carried out and the expressions are simplified by use of the Wang rules (1). After that the Wang algebra is dropped and the resulting polynomials are regarded as ordinary polynomials. The symbols A, B, C, etc., are interpreted as resistances of the associated branches. (In case there are no meshes we take $M=1$.)

For example in the network of Figure 1

$$\begin{aligned} M &= (A + B + K)(C + D + K) \\ &= AC + AD + AK + BC + BD + BK + CK + DK. \end{aligned}$$

We can take M_1 to be $(A+C)M$. Clearly then

$$(4) \quad R = \frac{ABC + ABD + ABK + ADK + ACD + BCD + BCK + CDK}{AC + AD + AK + BC + BD + BK + CK + DK}.$$

Now resistance and conductance are reciprocals; thus $aA=1$, $bB=1$, etc. If these relations are employed it is seen that (3) and (4) agree. It is of interest to obtain (4) directly from the network equations which express Kirchhoff's second law (the net potential difference around a mesh vanishes). Thus

$$\begin{aligned} E &= (A + C)I - A\, I_2 - C\, I_3, \\ 0 &= -A\, I + (A + B + K)I_2 - K\, I_3, \\ 0 &= -C\, I - K\, I_2 + (C + D + K)I_3. \end{aligned}$$

Here I_2 is a mesh current in mesh (A, B, K) and I_3 is a mesh current in mesh (C, D, K). It can be easily checked that M_1 is the determinant of these equations and that M is a principal minor determinant.

The determinant M expressed as a function of the resistances of the branches is termed the *complementary discriminant*. The following identity holds between the discriminants

$$(5) \qquad S(a, b, c, d, k) = abcdkM(a^{-1}, b^{-1}, c^{-1}, d^{-1}, k^{-1}).$$

An analogous relation holds for S_1 and M_1. It is easy to check that relation (5) holds for the network of Figure 1. Thus bcd is a term of S, and M has the complementary term AK.

We have seen that any symmetric determinant can be written as a discriminant S. By use of identity (5) it follows that a determinant may also be evaluated by the Wang algebra applied to the complementary discriminant M.

It is also of importance to have expressions for the potentials of other junctions of the network. If E_2 is the potential of junction 2 relative to junction 0, then the *voltage ratio* is

$$(6) \qquad \frac{E_2}{E} = \frac{S_1 \cap S_2}{S_1} = \frac{M_1 \cap M_2}{M_1}.$$

Here S_2 is the product of stars omitting junctions 0 and 2, and $S_1 \cap S_2$ means the terms common to the polynomials S_1 and S_2. Clearly

$$S_2 = (a + b)(a + c + k),$$

so

$$S_2 = ac + ak + ab + bc + bk,$$

and

$$\frac{E_2}{E} = \frac{ak + ab + bc + bk}{ab + bc + bk + ad + cd + dk + ak + ck}.$$

The derivation in terms of mesh discriminants M_1 and M_2 is analogous. Formulae (2a), (2b), and (6) are seen to furnish a complete solution to the network problem.

A *cotree* of the network of Figure 1 is a set of two branches which when removed leave no meshes. For example (A, K) is a cotree. It was proved by Kirchhoff in 1847 that the terms of the mesh discriminant M are simply the cotree products. Kirchhoff's work has been amplified by Franklin [17]. A *tree* of the network of Figure 1 is a set of three branches which connect all junctions. For example (b, c, d) is a tree. It was proved by Sylvester that the terms of the star discriminant S are simply the tree products. Sylvester termed S an *unisignant determinant* [7]. Sylvester's work was contemporaneous with that of Kirchhoff, but he did not seem to be aware of the connection with electric networks. Long afterwards, in an appendix to Maxwell's treatise, J. J. Thomson discussed the formation of S by tree products [8, p. 409]. Viewed in this light the Wang algebra is a systematic algebraic method for finding the trees or cotrees of a network.

2. **Limitation and extension of Wang algebra.** This paper gives an analysis of why the Wang method works. In particular the rules just stated will be proved. This analysis sheds new light on the properties of network discriminants; possibly this is of more importance than the proof of the rules.

To understand the limitations of the Wang trick we now consider systems more general than the classical networks treated by Kirchhoff. Such systems can arise by imposing constraints in addition to those imposed by Kirchhoff's laws. For instance, in the example considered above, suppose that the constraints $E_3 = 2E_2$ is assumed. This constraint could be achieved by the use of an ideal transformer having a two-to-one step-up ratio. The diagram of the network is shown in Figure 2. Thereby the potential difference from 3 to 0 is maintained twice that from 2 to 0. The current entering the coil at 2 is twice that leaving the coil at 3. Because of the constraint the variable E_3 may be eliminated from the network equations. Carrying out the elimination yields

$$\begin{aligned} I &= (a + b)E - (b + 2a)E_2, \\ 0 &= -(b + 2a)E + (b + d + 4a + 4c + k)E_2. \end{aligned}$$

The determinant $S(a, b, c, d, k)$ of these equations is again termed the discriminant. Evaluating this determinant

$$S = ab + bd + ad + 4ac + 4bc + ak + bk.$$

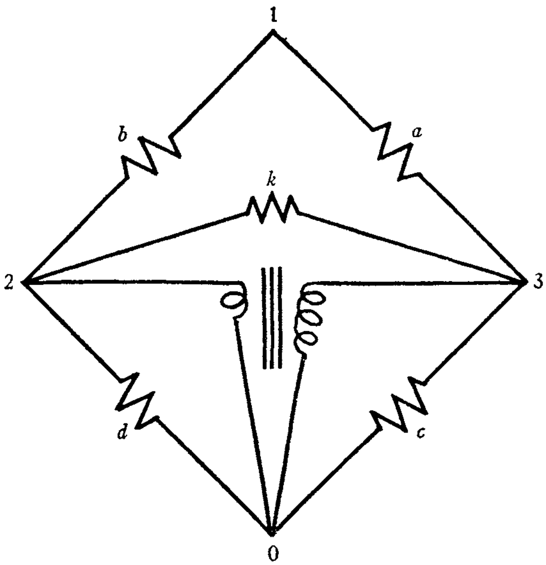

FIG. 2

Clearly S could not be directly given as a Wang product for the simple fact that the coefficients are not all the same. It follows that there must be a limitation to the Wang method.

A discriminant all of those coefficients are equal to one we term *just.* It was pointed out by H. W. Becker (see reference [14]) that there are just discriminants which do not come from classical Kirchhoffian networks. The question thus arises as to whether or not a just dsicriminant can be directly evaluated as a Wang product. An affirmative answer to this question is given by Theorem 1 and Theorem 2 to follow.

It is of interest to give an example of a non-Kirchhoffian network to which the Wang method applies. Consider a network whose branches are on the surface of a torus. The branches are not allowed to cross except at a junction. Then the network breaks up the surface of the torus into a number of regions. Such a network is shown in Figure 3. Two additional constraints are to be imposed on the current flow; no net current flow is permitted around the straight axis of the torus, and no net current flow is permitted around the circular axis of the torus. Then the current flow subject to this constraint can be achieved by taking mesh currents only around regions. Then the mesh discriminant M is obtained by taking the Wang product of all region meshes except one. In the example shown in Figure 3 there are seven meshes in the product.

This toroidal network is a special example of an interlinked electric and magnetic network [15]. Thus the two constraints could be achieved by mak-

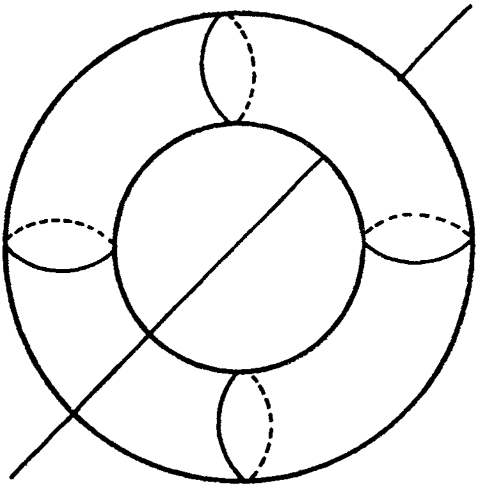

FIG. 3

ing the circular axis a core of infinite magnetic permeability. Likewise the straight axis would be a core of infinite magnetic permeability.

Before taking up the proof of Theorem 1 it is necessary to relate Wang algebra to the well-known Grassmann algebra. In fact, Wang algebra is simply the Grassmann algebra over the finite field mod 2. It will be shown that *any discriminant, just or not, can be evaluated by a trick like that of Wang but using Grassmann algebra instead of Wang algebra.*

3. **Grassmann algebra.** Let U be an n-dimensional vector space with scalar multipliers from a field F. Let the vectors $e_1, e_2, \cdots, e_n$ be a basis for U. Thus any vector u may be written

$$u = a_1e_1 + a_2e_2 + \cdots + a_ne_n.$$

The a_i are in F and are termed coordinates. This same basis will be used throughout in what follows. By the introduction of an outer product U will now be expanded into an algebra.

Let G be (2^n-1)-dimensional vector space with scalar multipliers from the field F. Let E_i, $i=1, 2, \cdots, 2^n-1$ be the basis for G. To convert G to an algebra the E_i are given a multiplication table. Thus E_i for $i=1, 2, \cdots, n$ serve as "generators" by the rules:

$$E_iE_j = - E_jE_i. \tag{7}$$

$$E_iE_i = 0. \tag{8}$$

With each integer i in the range $n<i<2^n$ we may associate a unique set of positive integers $i_1, i_2, \cdots, i_m$ where $m\geqq 2$ and $i_1<i_2< \cdots <i_m\leqq n$. This follows from the binomial theorem $(1+1)^n=\sum_0^n {}_nC_m$. Define

$$E_i = E_{i_1}E_{i_2} \cdots E_{i_m}. \tag{9}$$

It results from the rules (7) and (8) that the product of two or more of the generators gives 0 or $\pm E_i$ for some i in the range $n < i < 2^n$. Moreover, for any j and k, E_jE_k is 0 or $\pm E_i$. It is seen that this multiplication table is associative. Now any vectors in the vector space G may be multiplied to give another vector in G. Thus G is an algebra, a *Grassmann algebra.* (For the present purpose there is no need to adjoin a unit element.)

Returning to the space U we may identify e_i of U with E_i of G for $i = 1, \cdots, n$. Thus U may be thought of as generating the Grassmann algebra G. To distinguish the elements of U and G the elements of U may be referred to as vectors, and the elements of G may be referred to as *multivectors.*

Let $p_1, p_2, \cdots, p_m$ be m vectors of U. Of especial concern in this paper is the outer product of such a set

$$\pi = p_1p_2 \cdots p_m. \tag{10}$$

In the literature a multivector of type π is termed "simple" or "decomposable." General multivectors are not of concern here, so π is referred to simply as an *outer product* (even if $m = 1$). Each of the vectors p_i is expressed in terms of the basis, and the product in (10) is carried out and put in the *canonical form*

$$\pi = \sum d_ie_{i_1}e_{i_2} \cdots e_{i_m}; \qquad i_1 < i_2 < \cdots < i_m. \tag{11}$$

Here the $d_i \in F$ and are termed coefficients of the outer product. Naturally the coefficients do not depend on the order in which the product is carried out. Two outer products, π_1 and π_2, are said to be equivalent if $\pi_1 = c\pi_2$ where c is a nonzero scalar.

The first two lemmas to follow are standard theorems of Grassmann algebra; proof is supplied for the benefit of nonspecialists in geometry and also to introduce concepts and notations needed later.

LEMMA 1. *A necessary and sufficient condition that a set of vectors be dependent is that their outer product vanish.*

Proof. If $p_1, p_2, \cdots, p_m$ are dependent, then one can be expressed as a linear combination of the others, say $p_1 = a_2p_2 + a_3p_3 + \cdots + a_mp_m$. Thus

$$\begin{aligned} \pi &= (a_2p_2 + a_3p_3 + \cdots + a_mp_m)p_2p_3 \cdots p_m \\ &= a_2p_2^2p_3 \cdots p_m - a_3p_2p_3^2 \cdots p_m + \cdots . \end{aligned}$$

Clearly each term vanishes because of a repeated factor.

Turning to the second part of the lemma, we suppose that $p_1, p_2, \cdots, p_m$ are independent. Let

$$p_i = \sum_1^n k_{ij}e_j; \qquad k_{ij} \in F.$$

Some coordinate of p_1 does not vanish, say $k_{11} \neq 0$. Let $p_i' = p_i - b_i p_1$ for $i = 2, 3, \cdots, m$ where $b_i = k_{i1}/k_{11}$. Clearly $p_2', p_3', \cdots, p_m'$ are independent. The lemma is obviously true for $m = 1$, so suppose it is true for the case $m-1$. Then $\pi' = p_2' p_3' \cdots p_m' \neq 0$. Consider

$$p_1\pi' = k_{11}e_1\pi' + \sum_2^n k_{1j}e_j\pi'.$$

Now π' is a product of vectors which have no e_1 component; hence $e_1\pi' \neq 0$. For $j \geqq 2$ clearly $e_j\pi'$ can have no terms with e_1 as a factor. Hence $p_1\pi' \neq 0$. But

$$p_1\pi' = p_1(p_2 - b_2p_1)(p_3 - b_3p_1) \cdots = p_1p_2 \cdots p_m.$$

LEMMA 2. *Let $p_1, p_2, \cdots, p_m$ and $q_1, q_2, \cdots, q_k$ be two sets of vectors of U, both independent. Then these sets have equivalent outer products if and only if they are a basis for the same subspace of U.*

Proof. First suppose they span the same subspace. Take $m \geqq k$ and $p_i = \sum_1^k a_{ij}q_j$; $a_{ij} \in F$. Then

$$p_1p_2 \cdots p_m = (\sum a_{1j}q_j)(\sum a_{2j}q_j) \cdots \neq 0.$$

The q_i satisfy the same rules (7) and (8) that the e_i satisfy. Thus if the right side is multiplied out, each term would contain a repeated q_i factor if $m > k$, and so the whole expression would vanish; this is a contradiction. Thus $k = m$ and the right side reduces to $Aq_1q_2 \cdots q_m$ for some scalar $A \neq 0$.

Now suppose the outer products are equivalent; then $k = m$ and $p_1p_2 \cdots p_m = Aq_1q_2 \cdots q_m$ where $A \neq 0$. Let $v = \sum a_iq_i$. Then by Lemma 1, $vq_1q_2 \cdots q_m = 0$. Hence $vp_1p_2 \cdots p_m = 0$. Again by Lemma 1, $v = \sum b_ip_i$. Thus the two sets are a basis for the same subspace.

By virtue of Lemma 2 the subspaces of U and the inequivalent outer products are in one-to-one correspondence. Thus we may speak of "the outer product of a subspace"; there is no ambiguity except for a scalar factor. The following two lemmas are special results suited to the needs of §4.

LEMMA 3. *Let $\sum d_ie_{i_1}e_{i_2} \cdots e_{i_m}$ be a canonical form of the outer product for the subspace V.*

(a) *If $d_j = 0$ there is a nonzero vector r of V such that*

$$r = \sum a_ie_i \text{ with } a_{j_1} = a_{j_2} = \cdots = a_{j_m} = 0. \tag{12}$$

(b) *If $d_j \neq 0$ there is no such r in V. However, there is a set of vectors $r_1, r_2, \cdots, r_m$ with property* (12) *and such that*

$$q_1 = e_{j_1} + r_1,\ q_2 = e_{j_2} + r_2, \cdots, q_m = e_{j_m} + r_m$$

is a basis for V.

Proof. A basis of this type will be termed a *diagonal basis.* For example, if $m=3$ and $j_1=1$, $j_2=2$, and $j_3=3$, a diagonal basis has the form

$$\begin{array}{llllll} q_1 = e_1 & & & + a_4e_4 + & \cdots & + a_ne_n, \\ q_2 = & e_2 & & + b_4c_4 + & \cdots & + b_ne_n, \\ q_3 = & & e_3 & + c_4e_4 + & \cdots & + c_ne_n. \end{array}$$

To get on with the proof, formally replace all e_i by 0 except $e_{j_1}, e_{j_2}, \cdots, e_{j_m}$. In the space U this may be described as a projection of all of U into the m-dimensional subspace J with basis $e_{j_1}, e_{j_2}, \cdots, e_{j_m}$. In the space G this is a projection into a subspace G_J. Inspection of the fundamental definitions shows that G_J may be regarded as the Grassmann algebra over the space J. The projected elements will be distinguished by a prime. Let $p_1, p_2, \cdots, p_n$ be a basis for V. Then clearly $p_1' p_2' \cdots p_m' = d_j e_{j_1} e_{j_2} \cdots e_{j_m}$.

If $d_j=0$, then by Lemma 1 the p_i' are dependent, so $\sum b_i p_i' = 0$ and not all the b_i vanish. Let $r=\sum b_i p_i$, then $r \neq 0$ but $r'=0$. This proves the first part of the lemma.

If $d_j \neq 0$, it is clear by Lemma 1 that $r'=0$ implies all $b_i=0$. The p_i' are a basis for J, so $e_{j_1} = \sum b_{1_i} p_i'$, $e_{j_2} = \sum b_{2_i} p_i'$, etc. Thus $q_1 = \sum b_{1_i} p_i$ etc., is a diagonal basis.

LEMMA 4. *If a subspace V of U contains a vector $v=\sum c_i e_i$ and $c_s \neq 0$, then the outer product of V contains a nonvanishing term with e_s as a factor.*

Proof. Let $p_1, p_2, \cdots, p_m$ be a basis for V chosen so that $p_1=v$. It is sufficient to consider the case $e_s=e_1$. The second part of the proof of Lemma 1 then proves the statement of Lemma 4.

We now redefine the Wang algebra to be a Grassmann algebra over the finite field mod 2. Now $e_ie_j=-e_je_i=e_je_i$, so the algebra becomes commutative. If x is any element of the algebra $x+x=2x=0$. Let $x=\sum x_i$ where the x_i are monomials. Note that for a monomial in any Grassmann algebra $x_i^2=0$, so

$$x^2 = \sum x_i^2 + 2 \sum \sum x_i x_j = 0.$$

Thus x satisfies the rules (1), $x+x=0$ and $xx=0$.

The lemmas just proved hold, of course, for Wang algebra. There would have been simplification in the proofs if it had been assumed from the outset that F was the mod 2 field. However, this paper is also concerned with the Grassmann algebra when F is the real field.

For other applications and developments of Grassmann algebra, see Lichnerowicz [20], Bourbaki [21], S. MacLane [10], and Whitney [22]. A matrix theory related to Grassmann algebra is given by Wedderburn [11].

4. **The notion of a just subspace.** We say a vector is *just* if its coordinates are restricted to the values $+1$, -1, and 0. A subspace is said to be *just* if it has an outer product with coefficients restricted to the values $+1$, -1, or 0.

Such an outer product will be termed a just outer product. The following theorem will be basic to the analysis of Wang's method.

THEOREM 1. *Let V be an m-dimensional subspace of a real vector space U. A necessary and sufficient condition that V be a just subspace is that the just vectors of V constitute an m-dimensional vector space under addition* mod 2.

Proof. Let U_0 be the n-dimensional vector space over the finite field mod 2. Let V_s be the set of just vectors of V and suppose they constitute an m-dimensional vector space under addition mod 2. This means that each element of V_s has an image in U_0 obtained by neglecting the signs of the coordinates. (This may be a many-one correspondence.) Let V_0 be the image of V_s, then V_0 is an m-dimensional subspace of U_0. Let π_0 be an outer product of V_0 with coefficients d_i^0, and let π be an outer product of V with coefficients d_i. If $d_j^0=0$, then there is a vector r_0 which satisfies the conditions of Lemma 3a. Then r_0 is the image of a vector $r\in V_s$ which also satisfies the condition of this lemma. By Lemma 3b it follows that d_j nust be zero. If $d_j^0\neq 0$, there is a diagonal basis in V_0 relative to the jth term of π_0. Let this diagonal basis be the image of the set $p_1, p_2, \cdots, p_m$ in V_s. Clearly this set is independent and hence is a basis for V. Moreover, by a correct choice of signs, it may be supposed that this set is a diagonal basis relative to the jth term of π, so $d_j=1$ for this set, and clearly all the other coefficients of π have integral values. If $d_k^0\neq 0$, it follows in the same way that there is a diagonal basis $q_1, q_2, \cdots, q_m$ in V_s with an outer product having coefficients D_i and $D_k=1$. Then Lemma 2 gives $1=d_jD_k=d_kD_j$. Since d_k and D_j are integers, this implies that $D_j=1$ and $d_k=1$, or $D_j=-1$ and $d_k=-1$. Thus we have shown that the outer product of $p_1, p_2, \cdots, p_m$ has the property that $d_j^0=0$ implies $d_j=0$, and $d_j^0\neq 0$ implies $d_j=\pm 1$. Thus this is a just outer product and V is a just subspace.

To prove the second part of the theorem we need the following lemma.

LEMMA 5. *Let $p_1, p_2, \cdots, p_m$ be a diagonal basis having a just outer product. Then any subset of these vectors has a just outer product. In particular these vectors are just.*

Proof. Let the set $p_1, p_2, \cdots, p_m$ be a diagonal basis relative to the jth term of π. Let $\pi'=p_2p_3\cdots p_m$, so $\pi=p_1\pi'=e_{j_1}\pi'+r_1\pi'$. Then r_1 and π' have no factor e_{j_1} in their terms. Hence $e_{j_1}\pi'$ and $r_1\pi'$ have no terms in common. Then $e_{j_1}\pi'$ is just, and consequently π' is just because there can be no cancellation. By the definition of a diagonal basis it is apparent that any subset is a diagonal basis of lower dimension. Thus starting with π' we can perform a reduction by the same arguments, etc.

Returning to the proof of the theorem, suppose that V is a just subspace and let $p_1, p_2, \cdots, p_n$ be a diagonal basis for V. By Lemma 5 these vectors are just and $p_2, p_3, \cdots, p_m$ serve as a diagonal basis for a just subspace V'

of dimension $m-1$. The theorem is obviously true for a one-dimensional subspace, so by the inductive method suppose it is true for a $(m-1)$-dimensional subspace. In particular it thereby follows that there is a vector $y \in V_s'$ such that

$$y \equiv p_2 + p_3 + \cdots + p_m \bmod 2.$$

We seek a vector $x \in V_s$ such that

$$x \equiv p_1 + p_2 + \cdots + p_m \bmod 2. \tag{15}$$

Then $x_1 = p_1 \pm y$ satisfies this congruence. Suppose, however, that x_1 does not belong to V_s for either choice of sign. In any case the coordinates of x_1 are restricted to the values ± 2, ± 1, and 0. Thus if for one choice of sign a 2 occurs as a coefficient of e_s, then with the other choice a zero occurs. Suppose then the sign is chosen so that x_1 has no e_s component. By Lemma 4 the outer product of V contains a term $d_j e_{j_1} e_{j_2} \cdots e_{j_m}$ and $e_{j_h} = e_s$ and $d_j \neq 0$. Let $q_1, q_2, \cdots, q_m$ be a diagonal basis relative to this term. Then $x_1 = \sum_1^m b_i q_i$. By the property of a diagonal basis, all the b_i are integers, since x_1 has integer coordinates. Moreover, $b_h = 0$, since x_1 has no e_{j_h} component. Thus x_1 is in the $m-1$ dimensional subspace V'' with basis $q_1, q_2, \cdots, q_{h-1}, q_{h+1}, \cdots, q_m$. This subspace is just by Lemma 5. Now x_1 is congruent mod 2 to a set of the q_i since the b_i are integers. By the inductive hypothesis there is an $x \in V_s''$ such that $x \equiv x_1 \bmod 2$. This is the desired solution of (15). The same argument shows that there is an $x \in V_s$ congruent to the sum of any subset of the p_i. Thus

$$x \equiv \sum \delta_i p_i, \qquad \delta_i = 0 \quad \text{or} \quad \delta_i = 1. \tag{15a}$$

Now let u and v be two arbitrary elements of V_s. Then $u+v = \sum b_i p_i$ where the b_i are integers. Thus $u+v$ is congruent mod 2 to a sum of a subset of the p_i. We have shown that such a sum is congruent to a single element of V_s. This proves the group property under addition mod 2. As the set $p_1, p_2, \cdots, p_m$ is independent mod 2 and is a basis for this space, it follows that the space has dimension m.

5. **The discriminant of a subspace.** Let the vectors $p_1, p_2, \cdots, p_m$ be a basis for an m-dimensional subspace V of an n-dimensional vector space U. Let $k_{11}, k_{12}, \cdots$ be the coordinates of p_1, etc. Thus

$$p_i = \sum_1^n k_{ij} e_j. \tag{16}$$

The outer product of $p_1, p_2, \cdots, p_m$ has the canonical form

$$\pi = \sum d_i e_{i_1} e_{i_2} \cdots e_{i_m}. \tag{17}$$

Let K be the rectangular matrix with matrix elements k_{ij}. Let K^i be the square matrix consisting of the $i_1, i_2, \cdots, i_m$ columns of K in order. Then

$$d_i = \det K^i. \tag{18}$$

The derivation of (18) is clear from the definition of a determinant together with the projection method of proof used in Lemma 3.

Let $g_1, g_2, \cdots, g_n$ be indeterminate variables. Let G be an n by n diagonal matrix whose diagonal elements are $G_{ii}=g_i$. The matrix KGK_t may be regarded as an m by m matrix (K_t is the transpose of K). Let

$$f(g_1, g_2, \cdots, g_n) = \det KGK_t. \tag{19}$$

Then f is termed a *discriminant* of the matrix G in the subspace V. The discriminant is a multilinear form in the g_i and homogeneous of degree m in these variables. The following relation is a key theorem, stating that *a discriminant is a sort of inner product of the outer product with itself.*

$$f = \sum d_i^2 g_{i_1} g_{i_2} \cdots g_{i_m}. \tag{20}$$

The proof of (20) is given by noting that if all $g_i=0$ except $g_{j_1}, g_{j_2}, \cdots, g_{j_m}$, then (20) reduces to one term whose coefficient is $d_j^2 = \det K^j \det K^j_t$. Clearly the degree of f can not be greater than m, so this accounts for all the terms of (20). Of course there is no significance in the order of the g_i in the terms of f as the g_i are commutative. Comparing (17) and (20) and making use of Lemma 2 shows that *except for a constant factor the discriminant does not depend on the choice of the basis for* V.

A discriminant may be termed *just* if its coefficients are restricted to the values 1 or 0. The above considerations show that *a subspace is just if and only if it has a just discriminant.*

Several properties of the discriminant were studied in a paper by Raoul Bott and the writer [9]. (This paper will be referred to as B-D.) The discussion in B-D was restricted to real vector spaces, and the discriminant was arbitrarily normalized by the rule $f(1, 1, \cdots, 1)=1$. Because of theorem (20), the results of B-D imply certain properties of outer products. For example it was shown that two subspaces have the same discriminant if and only if one subspace is a reflection of the other in the coordinate planes. Then (20) gives

COROLLARY 1. *If one subspace is a reflection in the coordinate planes of another subspace, then they will have outer products whose coefficients differ only in sign. Conversely two outer products with this property correspond to reflected subspaces.*

Let V and V' be orthogonal complementary subspaces, i.e., V' consists of the vectors which are orthogonal to all the vectors of V. Let f and f' be normalized discriminants of V and V' respectively. It was shown in B-D that the following identity holds

$$f(g_1, g_2, \cdots, g_n) = g_1 g_2 \cdots g_n f'(g_1^{-1}, g_2^{-1}, \cdots, g_n^{-1}). \tag{5a}$$

This is essentially the same as relation (5) stated above. Clearly the following statement is a consequence of (5a).

COROLLARY 2. *If a subspace is just, its orthogonal complement is also just.*

The utility of the Wang algebra rests on the following result:

THEOREM 2. *Let V be a just, m-dimensional subspace of a real n-dimensional vector space U. Let $p_1, p_2, \cdots, p_m$ be a set of vectors of V which have integral coordinates and are independent* mod 2. *Then a discriminant of V is given by the Wang product $p_1p_2 \cdots p_m$ when in the canonical form e_i is identified with g_i.*

Proof. Consider the Grassmann algebra G_0 generated by the n-dimensional vector space U_0 over the finite field mod 2. This is the Wang algebra. In this algebra $p_1p_2 \cdots p_m$ does not vanish because of Lemma 1. The resulting outer product may be expressed with coefficients $d_i^0 = 0$ or 1. Since this is an algebra, the operation may be carried out in any order. Thus if no use is made of the mod 2 property, we obtain the ordinary outer product in G with coefficients d_i. Clearly the d_i are integral valued. Now $d_i \equiv d_i^0$ mod 2, so not all of the d_i vanish, and it follows that the set $p_1, p_2, \cdots, p_m$ is a basis for V. Since V is just, $d_i = d, -d$, or 0. Thus $d_i^0 = 0$ if and only if $d_i = 0$. Likewise $d_i^0 = 1$ if and only if $d_i = \pm d$. Comparing with (20) completes the proof.

Not every homogeneous multilinear form is a discriminant. The question arises as to the possibility of the generation of a new discriminant from two given discriminants f and F. If the variables are all different, it is easy to see that the product fF is again a discriminant and is just if f and F are just. The following theorem is a somewhat deeper result.

THEOREM 3. *Let $f(x_1, x_2, \cdots, x_n)$ and $F(y_1, y_2, \cdots, y_r)$ be discriminants. Let $f_i = \partial f/\partial x_i$, then for any i*

$$H(x_1, \cdots, x_n, y_2, \cdots, y_r) = f_i F(f/f_i, y_2, \cdots, y_r)$$

is a discriminant. Moreover, if f and F are just, then H is just.

Proof. We may write $F = Ay_1 + B$ where A and B do not contain y_1. Thus

$$H = f_i(Af/f_i + B) = Af + Bf_i.$$

In case f_i vanishes identically, we interpret the last expression as the definition of H. It is clear that H is a homogeneous multilinear form, or else it vanishes identically. Moreover, if f and F have coefficients 0 and 1, then H has coefficients 0 and 1.

We now employ Theorem 9 of B-D, which states that a necessary and sufficient condition that a homogeneous multilinear form $f(x_1, x_2, \cdots, x_n)$ be a discriminant is that $-\psi_{ij}$ be a square of a rational function. Here $\psi = \log f$ and $\psi_{ij} = \partial^2\psi/\partial x_i \partial x_j$. In applying this criteria to H it is necessary to consider the variables x and y separately. Let $\lambda = \log H$ and $\rho = \log F$. It is obvious that the criteria is satisfied for $\partial^2\lambda/\partial y_k \partial y_j$. Now consider $\partial^2\lambda/\partial y_k \partial x_j$.

$$\lambda = \rho + \log f_i$$
$$\partial\lambda/\partial y_k = \rho_k$$
$$-\partial^2\lambda/\partial y_k\partial x_j = \rho_{ki}\psi_{ij}/\psi_i^2.$$

The right side is clearly the square of a rational function. Now consider $\partial^2\lambda/\partial x_k\partial x_j$. We may write

$$H = A(f + f_iB/A) \quad \text{or} \quad H = Af(x_1. \cdots, x_i + B/A, \cdots, x_n).$$

It is obvious that the criteria is satisfied, since

$$\partial^2\lambda/\partial x_k\partial x_j = \psi_{kj}(x_1, \cdots, x_i + B/A, \cdots, x_n).$$

This completes the proof.

6. The subspaces of Kirchhoff. It is now appropriate to link the theory just developed to the network problem. In particular it is desired to prove the rules stated in §1.

In order to be precise, the following definitions are introduced: An electric network is a finite number of two-terminal *branches* arbitrarily interconnected at their terminal points. A point where one or more of these terminals meet is called a *junction*. A *direction* is arbitrarily assigned to each branch. A network is *connected* if any two of its junctions are connected by a chain of branches. Otherwise it consists of a number of disconnected *pieces*, each piece being a connected subnetwork. A *star* is the set of branches having one and only one terminal meeting at a given junction point. A *mesh* is a set of branches which form a simple closed chain. A *tree* is a connected set of branches (in each piece) which contain every junction but no meshes. A *cotree* is the complement of a tree.

Let a set of real numbers $u_1, u_2, \cdots, u_n$ be associated with the branches of a network. This set may be regarded as the coordinates of a vector relative to the basis of a real n-dimensional vector space U. Let a set of real numbers $Z_1, Z_2, \cdots, Z_J$ be associated with the junctions of a network. These are the *potentials* of the junctions. If the terminals of the branch i have potentials Z_a and Z_b, then $v_i = Z_a - Z_b$ is the *voltage* across the branch. Order a, b is determined by the preassigned direction of the branch. The voltages $v_1, v_2, \cdots, v_n$ define a vector v of V. The class of such vectors gives an m-dimensional subspace V which we term a *Kirchhoffian voltage space.*

Let $Z_i = 1$ and $Z_j = 0$ for $i \neq j$. This defines a *star vector* $p_i \in V$. Delete one star vector from each piece of the network. Then it is obvious that the remaining ones, say $p_1, p_2, \cdots, p_m$ form a basis for V. Then if $x \in V$

$$x = \sum a_i p_i. \tag{21}$$

Clearly each star vector is a just vector. The just vectors of V are a subset of the vectors obtained by giving the Z_i integral values. Thus if x is a just vector, the a_i must have integral values. Let $y = \sum b_i p_i$ be another just vector. Then if $a_e + b_e \not\equiv 0 \bmod 2$ etc.

$$x + y \equiv p_e + p_f + \cdots + p_h \mod 2. \tag{22}$$

The vector on the right corresponds to the junctions $e, f, \cdots, h$ being given the potentials 1 and all the other junctions the potential 0. Clearly this gives a just vector. Then the just vectors of V have the group property under addition mod 2, and since $p_1, p_2, \cdots, p_m$ are independent mod 2, it follows that the just vectors constitute an m-dimensional vector space under addition mod 2. Thus by virtue of Theorem 1 the *Kirchhoffian voltage space is just.*

Let v_i' be the current flowing through the ith branch of a network. By Kirchhoff's first law the net current entering a junction must vanish. Vectors of this class form a subspace V' of U which we term a *Kirchhoffian current space.* It is not difficult to show that V and V' are orthogonal complementary subspaces; this point is discussed in references [9; 16; 18]. It then follows from Corollary 2 that *the Kirchhoffian current space is just.*

It is of some interest to give a direct proof that V' is just. A *mesh vector* of V' is a unit flow of current confined to a mesh. Let x' be a just vector of V'. It is easy to see that

$$x' = w_1' + w_2' + \cdots + w_k' \tag{23}$$

where $w_1', w_2', \cdots, w_k'$ are mesh vectors, the corresponding meshes having no branches in common. Conversely any sum of this form gives a just vector x'. If two mesh vectors are added mod 2, it is clear that the sum is congruent to a sum of disjoint mesh vectors. Thus the just vectors of V' have the group property under addition mod 2.

Consider a tree of the network and add one other branch; the resulting network has exactly one mesh. Thus each branch not in the tree defines a different mesh. This method is employed by Ingram and Cramlet [12] and by Synge [13]. This gives mesh vectors $p_1', p_2', \cdots, p_r'$ which are a basis for V'. In fact, they are a diagonal basis and hence they are certainly independent mod 2. Thus the just vectors of V' constitute a vector space under addition mod 2, and this space has the same dimension as V'. The proof that the current space is just is completed by again using Theorem 1.

Next a proof will be given for formula (2a) which gives the resistance between the two junction points 1 and 0. Let $g_1, g_2, \cdots, g_n$ be the conductances of the branches of the network. Then since the voltage space is just, the discriminant $S(g_1, g_2, \cdots, g_n)$ is given by the Wang product of stars. To proceed it is at first supposed that there is a branch whose terminals are at 0 and 1. Let this be the branch g_i. Then according to equation (7) and Theorem 3 of B-D, the resistance across the branch g_1 is

$$R = \frac{1}{S}\,\frac{\partial S}{\partial g_i}\,. \tag{24}$$

Comparing this with (2a), we see that it remains to show $S_1 = \partial S/\partial g_1$. By the definition of S_1 we see that S_1 is independent of g_1 because g_1 only appears in the stars at junctions 1 and 0. Then from the Wang product expression for S we see $S=g_1S_1+P$ where P is also independent of g_1. This proves $S_1=\partial S/\partial g_1$, and it follows that (2a) and (24) are equivalent. By continuity it is clear that formula (2a) remains valid even when $g_1=0$, and the proof is complete.

To prove formula (2b), let $r_1, r_2, \cdots, r_n$ be the resistances of the branches of the network. Without loss of generality we may suppose that there is a branch connecting terminals 0 and 1 and that r_1 denotes its resistance. Let $M^*(r_1, \cdots, r_n)$ be the discriminant of the mesh space. Then as shown in B-D, the conductance G presented to a battery inserted in branch g_1 is

$$G = \frac{1}{M^*}\frac{\partial M^*}{\partial r_1}\cdot \tag{25}$$

It is seen that $M^*=M_1+Mr_1$ where M_1 and M are as defined in §2. Thus $G\equiv M/(M_1+Mr_1)$. Allowing r_1 to vanish, we have $R=1/G$, and formula (2b) is proved.

It remains to prove formula (6) for the voltage ratio. This is essentially the same as relation (33) of B-D. The signs of the terms are not determined by (33), but it is not difficult to show that no negative terms can occur in the present case.

The discriminant is defined as the determinant of KGK_t. For the star discriminant S it is seen that K is an incidence matrix between junctions and branches. For the mesh discriminant the matrix K is an incidence matrix between meshes and branches.

Now consider a matrix K with arbitrary integral matrix elements. It was shown in a previous paper [15] that such K matrices arise naturally in the analysis of transformer networks. Moreover, K has a simple topological interpretation as the *linkage matrix* between two networks, one electric and the other magnetic. The matrix element K_{ij} gives the number of times the ith mesh of the electric network links the jth mesh of the magnetic network. Here i and j run over a mesh basis in the networks. In particular any just discriminant can be realized in this way.

The writer is indebted to Raoul Bott for extensive discussions concerning the ideas in this paper; several of the problems treated here were formulated in this way. Acknowledgment is also due to H. W. Becker for use of his notes on Wang algebra.

References

1. K. T. Wang, *On a new method of analysis of electrical networks*, Memoir 2, National Research Institute of Engineering, Academia Sinica, 1934.

2. S. L. Ting, *On the general properties of electrical network determinants*, Chinese Journal of Physics vol. 1 (1935) pp. 18–40.

3. Chin-Tao Tsai, *Short cut methods for expanding the determinants involved in network problems*, Chinese Journal of Physics vol. 3 (1939) pp. 148–181.

4. Mong-Kang Ts'en, *Indicial impedances and admittances*, Chinese Journal of Physics vol. 4 (1940) pp. 89–96.

5. Wei-Liang Chow, *On electrical networks*, J. Chinese Math. Soc. (1940).

6. H. W. Becker, *Wang algebra*, 1948, unpublished.

7. T. Muir, *Contributions to the history of determinants* (1900–1920), London Blackie, 1930.

8. J. C. Maxwell, *A treatise on electricity and magnetism*, 3rd. ed., Oxford, Clarendon Press, 1891.

9. R. Bott and R. J. Duffin, *On the algebra of networks*, Trans. Amer. Math. Soc. vol. 74 (1953) pp. 99–109.

10. S. MacLane, *Grassman algebras and determinants*, mimeographed lectures, University of Chicago.

11. J. H. M. Wedderburn, *Lectures on matrices*, Amer. Math. Soc. Colloquium Publications vol. 17, 1934, p. 16.

12. W. H. Ingram and C. M. Cramlet, *On the foundations of electrical network theory*, J. Math. Phys. vol. 23 (1944) pp. 134–155.

13. J. L. Synge, *The fundamental theorem of electrical networks*, Quart. Appl. Math. vol. 9 (1951) pp. 113–127.

14. R. M. Cohn, *The resistance of an electrical network*, Proc. Amer. Math. Soc. vol. 1 (1950) pp. 316–324.

R. J. Duffin, *Nonlinear networks* IV, Proc. Amer. Math. Soc. vol. 1 (1950) pp. 233–240.

16. H. Weyl, *Repartición de corriente en una red eléctrica*, Rev. Mat. Hisp.-Amer. vol. 5 (1923) pp. 153–164.

17. Philip Franklin, *The electric currents in a network*, J. Math. Phys. vol. 4 (1925) pp. 97–102.

18. B. D. H. Tellegen, *A general network theorem, with applications*, Philips Res. Rep. vol. 7 (1952) pp. 259–269.

19. S. MacLane, *A combinatorial condition for planar graphs*, Fund. Math. vol. 28 (1936) p. 22.

20. A. Lichnerowicz, *Algèbre et analyse linéaires*, Paris, Masson, 1947.

21. N. Bourbaki, *Éléments de mathématique*, *Livre* II, *Algebre*, Chapitre III, Paris, Herman, 1948.

22. H. Whitney, *Geometric integration theory*, Princeton University Press, Princeton, 1957.

Carnegie Institute of Technology,
Pittsburgh, Pennsylvania