Transposing all terms to the left side produces the problem in the desired form; namely, find a vector $\mathbf{w}=(w_{11}, w_{12}, w_{13}, w_{14}, \cdots, w_{41}, w_{42}, w_{43}, w_{44})$ such that

$$B^t\mathbf{w} > 0,$$

where B^t is the matrix of coefficients of the above inequalities.

Networks Composed of Ternary Majority Elements

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Summary

Some methods of synthesis of networks composed of ternary majority elements such as the three-stable-state parametron or twin are studied. It is shown that three-place majority operation and diametrical negation (negation in the strong sense) with constants are functionally complete in three-valued logic. Classes of functions of two variables are studied for which networks realizing them contain no more than three cascades. This provides a possibility of realizing these functions in one full time unit in the case of three-cycle excitation. Methods of synthesis are developed for the networks which realize functions of an arbitrary number of variables. These methods are based on the excluding of one or several variables simultaneously. Considerations are proposed on the order of the excluding variables. Methods of majority decomposition of functions of three-valued logic are developed.

Introduction

At the present time many attempts to build ternary automatic and control systems are being made. The question arises whether such systems have some advantages in comparison with those of binary ones. The solution of this problem can not be provided without detailed analysis of ternary networks and a general comparison with binary ones.

Interest in ternary networks is no longer only academic since the appearance of three-stable-state elements which are equivalent in their complexity of construction to two-stable-state elements. In particular, a three-stable-state parametron is such an element.

The threshold properties of the parametron provide natural reasons for using it as a threshold element also in the three-stable-state case.

Properties of ternary threshold functions were studied in Bogolubov and Varshavsky [1], Hanson [2], and Merrill [3]. In many cases, however, a general threshold conception is technically unreasonable. From the practical point of view, it seems to be more convenient to have in design only identical elements with a limited number of inputs and equal weights of all inputs. Networks of such elements were studied in Varshavsky [4]. In the present paper we intend to continue these investigations.

1) The three-place operation on variables x, y, z which assume values -1, 0, 1 defined as

$$x \triangle y \triangle z = \begin{cases} 1 & \text{if } x+y+z \geq 1 \\ 0 & \text{if } x+y+z = 0 \\ -1 & \text{if } x+y+z \leq -1 \end{cases} \qquad (1)$$

we shall call a ternary majority operation. The operation of diametrical negation $\bar{x} = -x$ we shall call inversion.

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The ternary majority operation, negation, and constants 1 and 0 are functionally complete in three-valued logic. In order to prove completeness, it is sufficient to show that the functions $\max(x, y)$, $\min(x, y)$, and

$$f_\sigma(x) = \begin{cases} 1, & \text{if } x = \sigma \\ -1, & \text{if } x \neq \sigma \end{cases}$$

may be expressed in terms of $x\triangle y\triangle z$ and $\bar{x}$ (together with the constants) [5], [6]. Note that $\min(x, y) = \overline{\max(\bar{x}, \bar{y})}$.

Further,

a) $f_{-1}(x) = \bar{x} \triangle \bar{x} \triangle -1$

b) $f_1(x) = x \triangle x \triangle -1$

c) $f_0(x) = \overline{f_{-1}(x)} \triangle \overline{f_1(x)} \triangle -1$

d) $\max(x, y) = (\bar{x} \triangle y \triangle -1) \triangle x \triangle 1$.

The correctness of these expressions which can be established by substitution of different values of x and y proves the completeness.

2) Functions of one variable we shall write as a row

$$F(x) = \|F(-1); F(0); F(1)\|$$

and functions of two variables as a matrix

$$F(x_1, x_2) = \begin{Vmatrix} F(-1, & 1); F(0, & 1); F(1, & 1) \\ F(-1, & 0); F(0, & 0); F(1, & 0) \\ F(-1, & -1); F(0, & -1); F(1, & -1) \end{Vmatrix}.$$

An application of basic operations to the row or to the matrix is reduced to an application of these operations to every element of the row or of the matrix, respectively.

Now consider realizations of functions of one variable. Table I contains all $3^3=27$ functions of one variable.

Note that $f_{-a} = -f_a = \bar{f}_a$. Further, the majority operation $x\triangle y\triangle z$ is a self-dual function in three-valued logic, i.e.,

$$x_1 \triangle x_2 \triangle x_3 = \overline{\bar{x}_1 \triangle \bar{x}_2 \triangle \bar{x}_3}.$$

Hence, it is sufficient to consider the functions with positive indexes only because the networks realizing the functions with negative indexes may be obtained from the networks realizing the functions with positive indexes by means of inversion of all input variables and constants.

The functions f_0 and f_{13} are constants. The realizations of all other functions of one variable with positive indexes are shown in Fig. 1. so that the network of 11 functional elements produces all functions of one variable. (Where on Fig. 1 no connection is made, "0" is intended.)

Any function of two variables can be decomposed, as

$$F(x_1, x_2) = U_i^{-1}(x_1, x_2) \triangle U_j^0(x_1, x_2) \triangle U_k^1(x_1, x_2) \qquad (2)$$

where

$$U_i^{-1}(x_1, x_2) = \|f_i(x_2); 1; 1\|, \qquad U_j^0(x_1, x_2) = \|1; f_j(x_2); -1\|$$
$$U_k^1(x_1, x_2) = \|-1; -1; f_k(x_2)\|.$$

Note that $U_j^{-1}(x_1, x_2)$ is equivalent to $U_j^1(x_1, x_2)$ with respect to inversion of constants.

Consider the realizations of the functions $U_j^{-1}(x_1, x_2)$ with different indexes j. Among them are functions which are equivalent to each other with respect to inversion of x_2. The realizations of representatives of the functions of the type U_i^{-1} are given in Table II.

Thus, all functions of the types $U_i^{-1}(x_1, x_2)$ and $U_k^{-1}(x_1, x_2)$ are realizable by two-cascade networks which contain no more than four elements.

Now consider realizations of the different types of the $U_j^0(x_1, x_2)$. There are ten different types of the $U_j^0(x_1, x_2)$ exact up to inversion of variables and constants. Their realizations are given in Table III.

Hence, all types of the functions of two variables except possibly

$$A = \begin{Vmatrix} F(-1, \;\; 1) & 0 & F(1, \;\; 1) \\ 0 & -1 & 1 \\ F(-1, -1) & 1 & F(1, -1) \end{Vmatrix}$$

TABLE I

x	f_{-13}	f_{-12}	f_{-11}	f_{-10}	f_{-9}	f_{-8}	f_{-7}	f_{-6}	f_{-5}	f_{-4}	f_{-3}	f_{-2}	f_{-1}	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}
−1	−1	0	1	−1	0	1	−1	0	1	−1	0	1	−1	0	1	−1	0	1	−1	0	1	−1	0	1	−1	0	1
0	−1	−1	−1	0	0	0	1	1	1	−1	−1	−1	0	0	0	1	1	1	−1	−1	−1	0	0	0	1	1	1
1	−1	−1	−1	−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1

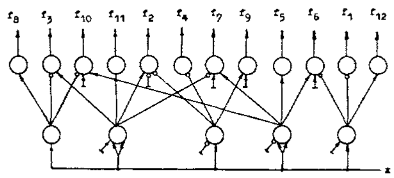

Fig. 1. Network generating all functions of one variable.

TABLE II

i	Realizations	Equivalent to
13	1	—
12	$(x_1 \triangle \bar{x}_2 \triangle 1) \triangle 1 \triangle 0$	4
11	$(x_1 \triangle \bar{x}_2 \triangle 1) \triangle (x_1 \triangle \bar{x}_2 \triangle 1) \triangle 1$	− 5
10	$(x_1 \triangle x_2 \triangle x_2) \triangle \bar{x}_2 \triangle 1$	—
9	$(x_1 \triangle 1 \triangle 0) \triangle (\bar{x}_2 \triangle -1 \triangle 0) \triangle 1$	1
8	$(x_1 \triangle 1 \triangle 0) \triangle (x_1 \triangle 1 \triangle 0) \triangle \bar{x}_2$	− 8
7	$(x_1 \triangle x_2 \triangle x_2) \triangle (x_1 \triangle \bar{x}_2 \triangle \bar{x}_2) \triangle 1$	—
6	$(x_1 \triangle x_2 \triangle 0) \triangle (x_1 \triangle \bar{x}_2 \triangle \bar{x}_2) \triangle 1$	− 2
5	$(x_1 \triangle 1 \triangle 0) \triangle (x_1 \triangle 1 \triangle 0) \triangle (\bar{x}_2 \triangle \bar{x}_2 \triangle -1)$	−11
3	$(x_1 \triangle 1 \triangle 0) \triangle (x_2 \triangle x_2 \triangle 1) \triangle \bar{x}_2$	—
2	$(x_1 \triangle x_1 \triangle 1) \triangle (\bar{x}_2 \triangle \bar{x}_2 \triangle 1) \triangle (x_2 \triangle 1 \triangle 0)$	− 6
0	$x_1 \triangle 1 \triangle 0$	—
− 1	$(x_1 \triangle x_1 \triangle 1) \triangle (\bar{x}_2 \triangle 1 \triangle 0) \triangle 0$	− 9
− 3	$(x_1 \triangle x_2 \triangle 0) \triangle (x_1 \triangle \bar{x}_2 \triangle 0) \triangle 1$	—
− 4	$(x_1 \triangle \bar{x}_2 \triangle 0) \triangle (x_1 \triangle x_1 \triangle 1) \triangle 1$	−12
7	$(x_1 \triangle x_1 \triangle 1) \triangle (x_2 \triangle x_2 \triangle 1) \triangle (\bar{x}_2 \triangle \bar{x}_2 \triangle 1)$	—
−10	$(x_1 \triangle x_1 \triangle 1) \triangle (x_2 \triangle \triangle x_2 \triangle 1) \triangle \bar{x}_2$	—
−13	$x_1 \triangle x_1 \triangle 1$	—

TABLE III

j	Realizations	Equivalent to
13	$\bar{x}_1 \triangle \bar{x}_1 \triangle 1$	−13
12	$(\bar{x}_2 \triangle 1 \triangle 0) \triangle \bar{x}_1 \triangle \bar{x}_1$	4, −4, −12
11	$(\bar{x}_2 \triangle \bar{x}_2 \triangle 1) \triangle \bar{x}_1 \triangle \bar{x}_1$	5, −5, −11
10	$(\bar{x}_2 \triangle \bar{x}_2 \triangle -1) \triangle (\bar{x}_1 \triangle \bar{x}_1 \triangle x_2) \triangle (\bar{x}_1 \triangle \bar{x}_1 \triangle 1)$	−10
9	$(\bar{x}_1 \triangle \bar{x}_1 \triangle 1) \triangle (\bar{x}_2 \triangle -1 \triangle 0) \triangle \bar{x}_1$	1, −1, −9
8	$\bar{x}_1 \triangle \bar{x}_1 \triangle \bar{x}_2$	−8
3	$(\bar{x}_1 \triangle \bar{x}_2 \triangle 1) \triangle (\bar{x}_1 \triangle x_2 \triangle 1) \triangle -1$	−3
0	$\bar{x}_1$	—
6	is not realizable by two-cascade network	2, −2, −6
7	is not realizable by two-cascade network	−7

and

$$B = \begin{Vmatrix} F(-1,1) & 1 & F(1,1) \\ F(-1,0) & -1 & 1 \\ F(-1,-1) & 1 & F(1,-1) \end{Vmatrix}$$

are realizable by networks which contain no more than 13 elements in three cascades.

Any function of the type A can be decomposed, as

$$A = \begin{Vmatrix} F(-1,\ 1) & 1 & 1 \\ 1 & 1 & 1 \\ F(-1,-1) & 1 & 1 \end{Vmatrix} \triangle \begin{Vmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & -1 \end{Vmatrix} \triangle \begin{Vmatrix} -1 & -1 & F(1,\ 1) \\ -1 & -1 & 1 \\ -1 & -1 & F(1,-1) \end{Vmatrix}$$

where the first and the third matrices were shown to be realizable by two-cascade networks and the second is presentable, as

$$\begin{Vmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & -1 \end{Vmatrix} = (\bar{x}_1 \triangle x_2 \triangle -1) \triangle (\bar{x}_2 \triangle \bar{x}_2 \triangle -1) \triangle (\bar{x}_1 \triangle \bar{x}_1 \triangle 1).$$

The realizations of some special kinds of the matrices of the type B may be suggested in the similar way. We could not find two-cascade realizations only for functions of the types:

$$\begin{Vmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{Vmatrix}; \quad \begin{Vmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{Vmatrix}; \quad \begin{Vmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{Vmatrix}.$$

The number of functions represented by these types is 18. It should be noted that the total number of the functions of two variables is $3^{3^2} = 19683$.

3) Now we consider methods of the synthesis of networks which realize functions of an arbitrary number of arguments. At first, we consider a method of synthesis by sequential excluding of variables.

To exclude one variable, we shall use the representation which we have already met in the synthesis of networks realizing functions of two variables. Every function of three-valued logic can be presented, as

$$\begin{aligned} F(x_1, Y) &= \|F(-1, Y); F(0, Y); F(1, Y)\| = \|F(-1, Y); 1; 1\| \\ &\quad \triangle \|1; F(0, Y); -1\| \triangle \|-1; -1; F(1, Y)\| \\ &= [(x_1 \triangle 1 \triangle 0) \triangle (x_1 \triangle 1 \triangle 0) \triangle F(-1, Y)] \\ &\quad \triangle [\bar{x}_1 \triangle \bar{x}_1 \triangle F(0, Y)] \\ &\quad \triangle [(x_1 \triangle -1 \triangle 0) \triangle (x_1 \triangle -1 \triangle 0) \triangle F(1, Y)]. \end{aligned} \qquad (3)$$

A network corresponding to (3) is presented in Fig. 2.

Now consider a joint co-excluding of two variables. Let it be necessary to exclude variables x_1 and x_2. We define other arguments of the function as Y. Functions which are obtained from $F(x_1, x_2, \cdots, X_n)$ by substitution of $x_1 = q_1$ and $x_2 = q_2$ into it we shall define as $F(q_1, q_2)$.

Function $F(x_1, x_2, Y)$ can be presented, as

$$\begin{aligned} F(x_1, x_2, Y) = & \left\| \begin{Vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & F(1,-1) \end{Vmatrix} \right\| \triangle \begin{Vmatrix} 1 & 1 & F(1,1) \\ 1 & F(1,1) & -1 \\ F(1,1) & -1 & -1 \end{Vmatrix} \\ & \triangle \begin{Vmatrix} 1 & F(1,0) & -1 \\ 1 & 1 & F(1,0) \\ 1 & 1 & 1 \end{Vmatrix} \\ & \triangle \begin{Vmatrix} F(-1,1) & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{Vmatrix} \triangle \begin{Vmatrix} -1 & -1 & -1 \\ F(-1,0) & -1 & -1 \\ 1 & F(-1,0) & -1 \end{Vmatrix} \\ & \triangle \begin{Vmatrix} 1 & 1 & F(-1,-1) \\ 1 & F(-1,-1) & -1 \\ F(-1,-1) & -1 & -1 \end{Vmatrix} \\ & \triangle \begin{Vmatrix} -1 & -1 & F(0,0) \\ -1 & F(0,0) & 1 \\ F(0,0) & 1 & 1 \end{Vmatrix} \triangle \begin{Vmatrix} -1 & F(0,1) & 1 \\ -1 & -1 & F(0,1) \\ -1 & -1 & -1 \end{Vmatrix} \\ & \triangle \begin{Vmatrix} 1 & 1 & 1 \\ F(0,-1) & 1 & 1 \\ -1 & F(0,-1) & 1 \end{Vmatrix}. \end{aligned} \qquad (4)$$

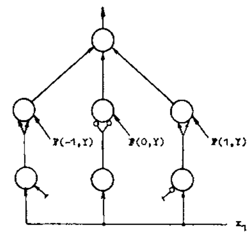

Fig. 2. Excluding of one variable.

A network corresponding to (4) is shown in Fig. 3. Not to overload the paper with a detailed proof, we give only a network by which three variables can be excluded (Fig. 4).

As for the realization of all functions of one variable, no more than 17 elements in two cascades are required, any function of four arguments is realizable by no more than 79 elements in six cascades, and any function of three arguments is realizable by no more than 39 elements in five cascades.

In most practical cases it is often possible to synthesize networks with a considerably lower number of elements. For example, the sum by modulo 3 of two variables is realized as

$$\begin{Vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{Vmatrix} = (x_1 \triangle x_2 \triangle 0) \triangle (\bar{x}_1 \triangle \bar{\bar{x}}_1 \triangle x_2) \triangle (\bar{x}_2 \triangle \bar{\bar{x}}_2 \triangle x_1).$$

In synthesizing, the order of excluding variables often appears to be essential. In synthesizing networks which realize functions of three and four variables, the number of different orders is relatively small, especially when two or three variables are simultaneously excluded; in synthesizing networks of higher complexity, however, the amount of enumeration rapidly increases.

For the choice of the order of excluding variables, it seems to be reasonable to use approximate methods which provide the choice of an optimal order only, that is, on the average over all the networks. We consider here one heuristic way. We have no proof for it, but, as we have shown, it provides considerable simplification of networks. The general idea of this method is the following: the first to be excluded are those variables that bear the greatest amount of information about the function.

The probabilistic scheme is associated with a function. The appearance of every constituent is of equal probability here. In this case every variable takes on any of its values with the probability 1/3. The entrophy of the function $F(x_1, x_2, \cdots, x_n)$ of three-valued logic can be calculated, as

$$H(F) = -\sum_j \frac{\sigma(T_j)}{3^n} \log_3 \frac{\sigma(T_j)}{3^n} \qquad j = -1, 0, 1,$$

where T_j is the set of n tuples of arguments where the function takes on the value j, and $\sigma(T_j)$ is the cardinal number of T_j.

The mean conditional entrophy of the function F with respect to its argument x_s can be calculated as

$$H(F/x_s) = -\frac{1}{3^n} \sum_i \sum_j \sigma(T_j \cap S_i) \log_3 \frac{\sigma(T_j \cap S_i)}{3^{n-1}}$$

where S_i is the set of n tuples of arguments with $x_s = i$, $i = -1, 0, 1$.

The amount of information the variable x_s contains about the value of the function F is

$$I(x_s, F) = H(F) - H(F/x_s).$$

These data can be calculated in the same way for both completely and incompletely defined functions.

4) For some functions networks realizing them can be synthesized by using methods of functional decomposition. A function $F(x_1, x_2, \cdots, x_n)$ is said to possess a disjunctive decomposition if it can be represented in the form

$$F(x_1, x_2, \cdots, x_n) = G[f(x_1, x_2, \cdots, x_k), x_{k+1}, x_{k+2}, \cdots, x_n] \qquad (5)$$

where $\{x_1, x_2, \cdots, x_k\}$ and $\{x_{k+1}, x_{k+2}, \cdots, x_n\}$ are nonoverlapping subsets of the set of arguments of the given function.

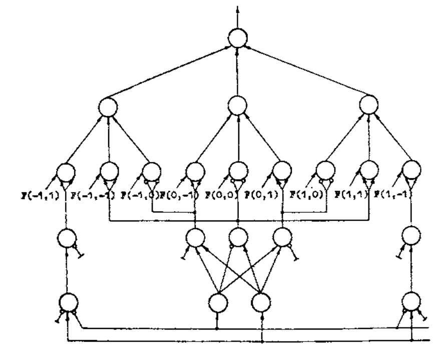

Fig. 3. Excluding of two variables.

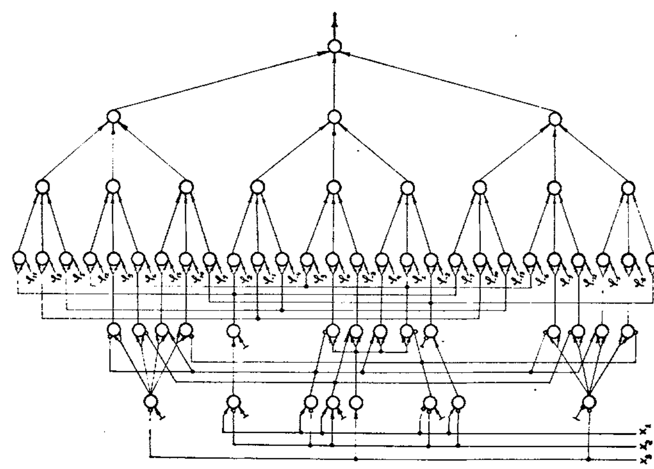

Fig. 4. Excluding of three variables.

Methods of the functional decomposition were studied by many authors [7]–[9]. In Varshavsky [10], necessary and sufficient conditions of decomposability of functions of three-valued logic have been developed. If the given function is decomposable, the representation of the function in form (5) considerably simplifies its realization. In this case, it is natural first of all to exclude a decomposition component.[1]

It is often convenient to reduce the problem of synthesis to solving a system of linear inequalities. In this case, the desirable configuration of the network may be chosen in advance.

Consider this approach in the example of synthesizing the summator modulo 3 for three variables. The function is defined by Table IV. We shall search the realization of this function in the form

$$F(x_1, x_2, x_3, x_4) = f_1 \triangle f_2 \triangle f_3 \qquad (6)$$

where

$$f_1 = f_1(m_0, m_1)$$
$$f_2 = f_2(m_0, m_2)$$
$$f_3 = f_3(m_0, m_3)$$

and

$$m_0 = x_1 \triangle x_2 \triangle x_3$$
$$m_1 = \bar{x}_1 \triangle x_2 \triangle x_3$$
$$m_2 = x_1 \triangle \bar{x}_2 \triangle x_3$$
$$m_3 = x_1 \triangle x_2 \triangle \bar{x}_3.$$

Expression (6) may be considered as an equation where f_1, f_2, and f_3 are unknown.

Define the values of the functions f_1, f_2, and f_3 are tuples of fixed values of m_0, m_1, m_2, and m_3, as

[1] The decomposition component is $f(x_1, x_2, \cdots, x_k)$ in (5).

TABLE IV

x_3 \ x_1, x_2	$x_1=-1$, $x_2=-1$	$x_1=0$, $x_2=-1$	$x_1=1$, $x_2=-1$	$x_1=-1$, $x_2=0$	$x_1=0$, $x_2=0$	$x_1=1$, $x_2=0$	$x_1=-1$, $x_2=1$	$x_1=0$, $x_2=1$	$x_1=1$, $x_2=1$
−1	0	1	−1	1	−1	0	−1	0	1
0	1	−1	0	−1	0	1	0	1	−1
1	−1	0	1	0	1	−1	1	−1	0

$$f_1(m_0, m_1) = \left\| \begin{matrix} a_2 & a_3 & a_4 \\ a_{-1} & a_0 & a_1 \\ a_{-4} & a_{-3} & a_{-2} \end{matrix} \right\|$$

$$f_2(m_0, m_2) = \left\| \begin{matrix} b_2 & b_3 & b_4 \\ b_{-1} & b_0 & b_1 \\ b_{-4} & b_{-3} & b_{-2} \end{matrix} \right\|$$

$$f_3(m_0, m_3) = \left\| \begin{matrix} c_2 & c_3 & c_4 \\ c_{-1} & c_0 & c_1 \\ c_{-4} & c_{-3} & c_{-2} \end{matrix} \right\|.$$

Developing (6) for every tuple of the fixed values of x_1, x_2, x_3, and x_4, we obtain the system of logical equations. This system, taking into account the definition of ternary majority operation, is equivalent to the following system of linear inequalities:

$$\left.\begin{aligned}
a_{-4} + b_{-4} + c_{-4} &= -0 \\
a_{-4} + b_{-1} + c_{-1} &\geq 1 \\
a_{-4} + b_2 + c_2 &\leq -1 \\
a_{-1} + b_{-4} + c_{-1} &\geq 1 \\
a_{-4} + b_{-4} + c_2 &\leq -1 \\
a_2 + b_{-4} + c_2 &\leq -1 \\
a_{-1} + b_{-1} + c_{-4} &\geq 1 \\
a_{-4} + b_2 + c_{-4} &\leq -1 \\
a_2 + b_{-4} + c_{-4} &\leq -1 \\
a_2 + b_2 + c_{-4} &\leq -1 \\
a_{-3} + b_0 + c_3 &= 0 \\
a_0 + b_{-3} + c_3 &= 0 \\
a_{-3} + b_3 + c_0 &= 0 \\
a_0 + b_0 + c_0 &= 0 \\
a_3 + b_{-3} + c_0 &= 0 \\
a_0 + b_3 + c_{-3} &= 0 \\
a_3 + b_0 + c_{-3} &= 0 \\
a_{-2} + b_0 + c_4 &\geq 1 \\
a_{-2} + b_4 + c_4 &\geq 1 \\
a_4 + b_{-2} + c_4 &\geq 1 \\
a_1 + b_1 + c_4 &\leq -1 \\
a_{-2} + b_4 + c_{-2} &\geq 1 \\
a_4 + b_4 + c_{-2} &\geq 1 \\
a_1 + b_4 + c_1 &\leq -1 \\
a_4 + b_{-2} + c_{-2} &\geq 1 \\
a_4 + b_1 + c_4 &\leq -1 \\
a_4 + b_4 + c_4 &= 0
\end{aligned}\right\} \qquad (7)$$

The system (7) has the following solution:

$$\begin{aligned}
a_{-4} &= b_{-4} = c_{-4} = a_0 = b_0 = c_0 = a_4 = b_4 = c_4 = 0, \\
a_{-1} &= b_{-1} = c_{-1} = a_{-2} = b_{-2} = c_{-2} = a_{-3} = b_{-3} = c_{-3} = 1 \\
a_1 &= b_1 = c_1 = a_2 = b_2 = c_2 = a_3 = b_3 = c_3 = -1;
\end{aligned}$$

then

$$\begin{aligned}
f_1 &= (m_0 \triangle \bar{m}_1 \triangle 0) \triangle (m_1 \triangle m_1 \triangle \bar{m}_0) \triangle (\bar{m}_1 \triangle \bar{m}_1 \triangle \bar{m}_0) \\
f_2 &= (m_0 \triangle \bar{m}_2 \triangle 0) \triangle (m_2 \triangle m_2 \triangle \bar{m}_0) \triangle (\bar{m}_2 \triangle \bar{m}_2 \triangle \bar{m}_0) \\
f_3 &= (m_0 \triangle \bar{m}_3 \triangle 0) \triangle (m_3 \triangle m_3 \triangle \bar{m}_0) \triangle (\bar{m}_3 \triangle \bar{m}_3 \triangle \bar{m}_0)
\end{aligned}$$

and the final network requires 17 elements in four cascades.

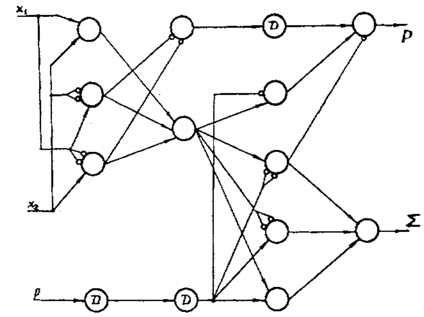

Fig. 5. Summator modulo 3.

It should be noted that a summator realizing the sum modulo 3 of three variables and generating a carryover may be designed of 11 elements and three delays using a number of informal considerations (Fig. 5).

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"Ideally Fast" Decimal Counters with Bistables

I. BRČIĆ

I. SUMMARY

The counting speed is taken as the most important factor in the selection of decimal counter logical arrangements. Decimal counter arrangements with the counting speed equal to that of the basic bistable, termed "ideally fast," are considered. The methods for making "ideally fast" counting logics are outlined. A number of "ideally fast" decimal counter arrangements with weighted and unweighted codes are presented and their features are discussed.

II. INTRODUCTION

To perform aperiodic counting in a scale of N, it is necessary to have a device or circuit which is capable of existing in any one of N different stable states and which at the occurrence of an input pulse steps from one state to the next in a cyclical sequence. A set of M bistable circuits assembled with decision elements in such a manner that the set as a whole can exist in $N \leq 2^M$ stable states is usually used when the counting is performed by means of circuits. A decimal

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