DECOMPOSITION OF ASYNCHRONOUS LOGICAL CIRCUITS. I

A. N. Chebotarev

The decomposition problems considered in the present paper are connected with the realization of automata by asynchronous logical circuits [1] with the presence of constraints on the classes of Boolean functions* on the right sides of the equations of the asynchronous logical circuits.

In [2], necessary and sufficient conditions for the realizability of an automaton by an asynchronous logical circuit were formulated and an algorithm for the design of the canonical realization was provided; however, no constraints were imposed on the classes of the Boolean functions obtained on the right sides of the equations. Since to each equation with a nontrivial right side there corresponds a logical element of the physical circuit, then, with the presence of constraints on the type of logical elements utilized, it follows that a corresponding constraint should be imposed also on the classes of Boolean functions on the right sides of the equations of the asynchronous logical circuits.

One of the possible approaches to the design of a circuit with the aforementioned constraints amounts to the following: A circuit with arbitrary Boolean functions on the right sides of the equations is initially constructed and, thereafter, a decomposition of each of the equations is performed in such fashion that all of the Boolean functions on the right sides of the equations are referred to the specified class. Such a decomposition of the equations is connected with the representation of Boolean functions in the form of compositions of functions of the specified classes.

It is obvious that the automaton realized by the circuit finally obtained does not coincide with the automaton realized by the original circuit for every decomposition of the equations. Therefore, the problem arises as to how to determine the class of decompositions not leading to changes of the circuit of the automaton to be realized and permitting one to obtain functions of the specified classes in the right sides of the equations.

In the present paper, this problem is solved for certain forms of decomposition of the so-called semi-modular circuits.

We have retained the terminology and notation adopted in [1].

BASIC DEFINITIONS

Definition 1. An asynchronous logical circuit S is a triple \( \langle Z, Z_0, F \rangle \), where \( Z = \{ z_1, z_2, \ldots, z_n \} \) is a finite set of binary variables, called the set of variables of the circuit; \( Z_0 = \{ z_1, z_2, \ldots, z_m \}, 0 \leq m \leq n \), is a subset of set \( Z \), called the set of input variables of the circuit; and \( F \) is a system of Boolean equations of the form \( z_i = f_i(z_1, z_2, \ldots, z_n), i = m + 1, \ldots, n \).

If we assume that \( Z \) is the set of all variables occurring on the left and also on the right sides of \( F \), while \( Z_0 \) is the set consisting of all those (and only those) variables of \( Z \) which do not occur on the left sides of equations, then the system of equations \( F \) will unambiguously determine circuit \( S \). Therefore, we shall henceforth identify circuit \( S \) with equation system \( F \).

Definition 2. A circuit \( S \) with an identified initial state \( p_i \) is called an initial circuit. We shall specify an initial circuit in the form of ordered pairs \( \langle S, p_i \rangle \) or \( \langle F, p_i \rangle \). We shall define the set of paths admissible for individual circuit \( \langle S, p_i \rangle \) as the set composed of all paths admissible for circuit \( S \) and beginning in state \( p_i \).

Definition 3. Initial circuits \( \langle F_1, p_i \rangle \) and \( \langle F_2, p_i \rangle \) are equivalent with respect to set of variables \( Z' \) if, for each variable \( z \in Z' \), the set of all paths of this variable, admissible for circuit \( \langle F_1, p_i \rangle \), coincides with the set of all its paths admissible for circuit \( \langle F_2, p_i \rangle \).

Definition 4. Two paths are strictly equivalent with respect to set of variables \( Z' \) if one and the same path vector \( Z' \) corresponds to them.

*The concept of a class of Boolean functions is used here in the sense of [3] (p. 198).


0011-4235/78/1402-0155 $07.50 ©1978 Plenum Publishing Corporation 155
Definition 5. The system of equations
\[ z_i = g_i(z_1, z_2, \ldots, z_n), \quad z_i = f_i(z_1, z_2, \ldots, z_n) \]
is called a simple decomposition of equations \( z_i = f_i(z_1, z_2, \ldots, z_n) \) if \( g_i(z_1, z_2, \ldots, z_n, f_i(z_1, \ldots, z_n)) = f_i(z_1, z_2, \ldots, z_n) \).

Definition 6. System of equations \( G_i \) is called a decomposition of the equations \( z_i = f_i(z_1, z_2, \ldots, z_n) \) if there exists a sequence of systems of equations \( G_i, G_{i+1}, \ldots, G_k \) such that \( G_i \) consists of the sole equation \( z_i = f_i(z_1, z_2, \ldots, z_n) \) and, for each \( j = 2, 3, \ldots, k \), \( G_j \) is obtained as the result of replacing some equation of \( G_{j-1} \) by its simple decomposition.

We shall call circuit \( F' \), obtained from circuit \( F \) by replacing one or several equations by their decompositions, a decomposition of circuit \( F \).

If circuit \( F' \) is obtained as the result of replacing in \( F \) only the equation \( z_i = f_i(z_1, z_2, \ldots, z_n) \) by its decomposition, then we shall call such a circuit the decomposition of circuit \( F \) by its \( i \)-th variable, denoting it either by \( F^{z_i} \) or simply by \( F^i \).

A decomposition of an initial circuit is an initial circuit.

It is possible to show that if \( F' \) is a decomposition of circuit \( F \), then, for each state \( p \) of circuit \( F \), there exists a unique state \( p' \) of circuit \( F' \), equivalent to it with respect to \( Z \) and such that
\[ W(p) = W'(p'). \quad (1) \]

As the initial state of decomposition of initial circuit \( <F, p> \) we adopt state \( p' \), satisfying condition (1).

**Theorem of Parallel Decompositions**

Let \( <F_i, p_i> \) and \( <F_j, p_j> \) be the decompositions of initial circuit \( <F, p> \) corresponding to the \( i \)-th and \( j \)-th variables, while \( <F_{ij}, p_{ij}> \) is the decomposition of this same circuit by both variables.

**Theorem 1.** If \( <F_i, p_i> \) and \( <F_j, p_j> \) are strictly equivalent to circuit \( <F, p> \) with respect to the set of variables \( Z \) (the set of variables of circuit \( F \)), then \( <F_{ij}, p_{ij}> \) is also strictly equivalent to \( <F, p> \) with respect to \( Z \).

Before we turn to the proof of this theorem, we shall formulate two lemmas.

**Lemma 1.** For each path admissible for circuit \( <F, p> \) there exists a path, strictly equivalent to it with respect to \( Z \), which is admissible for circuit \( <F_i, p_i> \).

**Lemma 2.** If circuit \( <F_i, p_i> \) is strictly equivalent to circuit \( <F, p> \) with respect to the set of variables \( Z \), then, for each state \( p_k \) lying on a path which is admissible for circuit \( <F_i, p_i> \), it is true that \( W'(p_k) \cap Z \subseteq W(p_k) \), where \( p_k = prZ(p_k) \).

The proofs of these lemmas are uncomplicated and thus we shall omit them here.

Let \( l = p_1, p_2, \ldots, p_s \) be an arbitrary path. By \( w_k \) we denote the set of all variables whose values are distinguished in states \( p_k \) and \( p_{k+1} \) (\( 1 \leq k < s \)).

Assertion 1. Let \( l \) be admissible for circuit \( F \) if and only if, for each \( j = 1, 2, \ldots, s - 1, w_j \subseteq W(p_j) \).

Proof of Theorem 1. As follows from Lemma 1, for each path admissible for circuit \( <F, p> \) there exists a path strictly equivalent to it with respect to \( Z \) and admissible for circuit \( <F_{ij}, p_{ij}> \). Therefore, for the proof of the theorem it suffices to prove the converse assertion.

Let \( l^{ij} = p_1^{ij}, p_2^{ij}, \ldots, p_s^{ij} \) be an arbitrary path which is admissible for circuit \( <F_{ij}, p_{ij}> \). We construct the sequence of states \( p_1, p_2, \ldots, p_s, p_{i}^{1j}, p_{i}^{2j}, \ldots, p_{i}^{lj}, p_{i}^{pj}, p_{i}^{pj}, \ldots, p_{i}^{pj} \) corresponding to circuits \( F_i, F_i^j, \) and \( F \) such that, for any \( k (1 \leq k \leq s) \),
\[ p_k = prZ(p_k), \quad p_k = prZ(p_k), \quad p_k = prZ(p_k). \quad (2) \]

If, in these sequences, from each group of consecutive identical states we remove all the states but one, we then obtain a path strictly equivalent to path \( l^{ij} \) with respect to the sets \( Z_i, Z_{ij}, \) and \( Z \).

We now show that the paths thus obtained are admissible for the corresponding circuits.

By \( w_k \) we denote the set of all variables whose values are distinguished in the \( k \)-th and \( (k + 1) \)-st states of sequence \( p_1^{ij}, \ldots, p_s^{ij} \). Then, in accordance with Assertion 1, it suffices to demonstrate that, for any \( k (1 \leq k \leq s) \),
We perform the proof by induction on \( k \). It is obvious that \( w^k_1 = w^1_k \cap Z^j_1, w^k_1 = w^1_k \cap Z^j, w_k = w^1_k \cap Z \), so that, when \( k = 1 \), the validity of assertion (3) follows from the definition of the initial state of the decomposition, according to which \( W^i(p^1_j) = W^i(p^1_j) = W(p_j) \).

Let assertion (3) have been proved for all \( k < n \). We now show that it is also true when \( k = n \).

We remark, first of all, that states \( p^1_n, p^1_i \), and \( p_n \) belong to paths which are admissible for the corresponding circuits.

We consider the following four cases.

1. \( z_i, z_j \notin w_n \); consequently, \( z_i, z_j \notin w_n \) etc. In this case, the validity of assertion (3) follows directly from (2) and from the fact that, for all variables different from \( z_i \) and \( z_j \), the equations in systems \( F^1_i, F^1 \), and \( F \) coincide with the corresponding equations of system \( F^1_i \).

2. \( z_i \in w_n \) and \( z_j \notin w_n \). Since the equations for variable \( z_i \) in systems \( F^1_j \) and \( F^1 \) coincide, \( z_i \in W(p^1) \); however, since circuit \( (F^1, p^1) \) is strictly equivalent to circuit \( (F, p^1) \), then, on the basis of Lemma 2, \( z_i \in W(p_n) \). The equations for variable \( z_i \) in systems \( F^1 \) and \( F \) coincide; thus, consequently, \( z_i \in W(p^1_n) \). For all the remaining variables of \( w^1_n \) the previous reasoning remains in force.

3. \( z_j \in w_n \), \( z_j \notin w_n \).

4. \( z_i \in w_n, z_j \notin w_n \).

The proofs for cases 3 and 4 are analogous to the proof for case 2.

The proof of Theorem 1 is thus completed.

It should be mentioned that the analogous theorem does not hold for equivalence. Theorem 1 also will be untrue if, instead of set \( Z \), we arbitrarily choose one of its subsets \( Z' \subset Z \).

Some Decomposition Classes

Definition 7. A decomposition of equation \( z_i = f_i(z_1, z_2, \ldots, z_n) \) of the form

\[
\begin{align*}
z_i &= g_i(z_1, z_2, \ldots, z_n) \lor z_i, \\
\end{align*}
\]

will be called a simple disjunctive decomposition.

A decomposition of equation \( z_i = f_i(z_1, z_2, \ldots, z_n) \), obtained as the result of successive performances of simple disjunctive decompositions, is called a disjunctive decomposition of this equation. We shall correspondingly speak of simple disjunctive decompositions and of disjunctive decompositions of circuit \( (F, p_1) \) by the \( i \)-th variable.

Definition 8. A decomposition of equation \( z_i = f_i(z_1, z_2, \ldots, z_n) \) of the form

\[
\begin{align*}
\begin{cases}
z_i = g_i(z_1, z_2, \ldots, z_n) \land z_i, \\
z_i = f_i(z_1, z_2, \ldots, z_n)
\end{cases}
\end{align*}
\]

is called a simple conjunctive decomposition.

The concept of a conjunctive decomposition is defined analogously to that of a disjunctive decomposition.

Definition 9. A decomposition of the equation \( z_i = f_i(z_1, z_2, \ldots, z_n) \) of the form

\[
\begin{align*}
\begin{cases}
z_i = \tilde{z}_i, \\
z_i = \tilde{f}_i(z_1, z_2, \ldots, z_n)
\end{cases}
\end{align*}
\]

is called an inverse decomposition.

DECOMPOSITIONS OF SEMIMODULAR CIRCUITS

Definition 10. Circuit \( (F, p_1) \) is called semimodular with respect to variable \( z \) if, for any pair of states \( p' \) and \( p'' \) which are attainable from \( p_1 \) such that \( p'' \in I(p', \phi(p')) \), it follows from \( z \in W(p') \) that one of the following two assertions is valid: 1) \( z \in W(p'') \); 2) the value of variable \( z \) in state \( p'' \) is opposite to its value in state \( p' \).
Circuit \( \langle F, p_i \rangle \) is called semimodular with respect to set \( Z' \) of variables if it is semimodular with respect to each variable \( z \in Z' \).

We shall call a circuit which is semimodular with respect to all its variables simply semimodular.

**Simple Disjunctive Decomposition**

Let \( \langle F^i, p^i \rangle \) be a simple disjunctive decomposition of the semimodular circuit \( \langle F, p_i \rangle \) with respect to the \( i \)-th variable.

**THEOREM 2.** Circuit \( \langle F^i, p^i \rangle \) is strictly equivalent to circuit \( \langle F, p_i \rangle \) with respect to the set of variables \( Z \) if and only if none of the paths admissible for circuit \( \langle F, p_i \rangle \) contains a segment \( p^i_{r_1} \ldots p^i_{r_s} \) with the following properties:

- a) The value of variable \( z_i \) in state \( p^i_{r_s} \) equals 1;
- b) \( f_i(p_{r_1}) = 0, f_i(p_{r_s}) = 1; \)
- c) \( g_i(p_{r_s}) = 0; \)
- d) for each \( r \) \((1 < r \leq s)\), from the fact that the value of \( z_i \) in state \( p_{r_s} \) equals 1, while \( f_i(p_{r_r}) = 1 \), it follows that \( g_i(p_{r_r}) = 1 \).

**Proof.** Necessity. We now show that from the existence of a path, admissible for circuit \( \langle F, p_i \rangle \), containing a segment \( p^i_{r_1} \ldots p^i_{r_s} \) with properties a-d, it follows that circuits \( \langle F, p_i \rangle \) and \( \langle F^i, p^i \rangle \) are not strictly equivalent, i.e., that we can find a path, admissible for \( \langle F^i, p^i \rangle \), which does not correspond to any path which is admissible for \( \langle F, p_i \rangle \). For this, we construct two paths, strictly equivalent with respect to \( Z \), \( p_1 \ldots p_s \) and \( p^i_1 \ldots p^i_k \), respectively, admissible for circuits \( \langle F, p_i \rangle \) and \( \langle F^i, p^i \rangle \) and such that \( z_i \in W^i(p^i_k) \) and \( z_i \notin W(p_s) \).

If we now choose the state which differs from \( p^i_k \) by the value of variable \( z_i \) as state \( p^i_{k+1} \), then we obtain a path, admissible for circuit \( \langle F^i, p^i \rangle \), which does not correspond to any path admissible for circuit \( \langle F, p_i \rangle \).

Let path \( p_1p_2 \ldots p_s \) be terminated by segment \( p_{r_1} \ldots p_{r_s} \) with properties a-d. We now show that it is possible to construct a path, strictly equivalent to it with respect to \( z_i \), which is admissible for circuit \( \langle F^i, p^i \rangle \) and is such that \( z_i = 0 \) in state \( p^i_{k+1} \) [consequently, \( z_i \in W^i(p^i_k) \) and \( z_i \notin W(p_s) \)].

We partition path \( p_1p_2 \ldots p_s \) into two parts \( p_1 \ldots p_m \) and \( p_{m+1} \ldots p_s \) in such a fashion that, beginning with state \( p_{m+1} \) and up to state \( p_s \) inclusive, function \( f_i \) equals 1.

Let \( p^i_1 \ldots p^i_k \ldots p^i_{k+1} \ldots p^i_{k+m} \ldots p^i_{m} \ldots p^i_{m+1} \ldots p^i_{m+s} \ldots p^i_{s} \) be a path which is admissible for circuit \( \langle F^i, p^i \rangle \) and also strictly equivalent to path \( p_1p_2 \ldots p_m \) with respect to \( Z \). We can always assume that \( z_i = 0 \) in state \( p^i_k \). If this were not the case, then we could adjoin to segment \( p^i_{r_1} \ldots p^i_{r_s} \) one more state differing from \( p^i_k \) only in its value of variable \( z_i \). Since \( f_i(p^i_k) = 0 \), the path we have obtained will be admissible for circuit \( \langle F^i, p^i \rangle \) and will also be strictly equivalent to path \( p_1p_2 \ldots p_m \) with respect to \( Z \).

As before, by \( w_j \) we denote the set of all those variables whose values differ as between states \( p_j \) and \( p_{j+1} \).

We shall extend path \( p^i_1 \ldots p^i_k \ldots p^i_{k+s} \) by setting, for all \( j = 0, 1, \ldots, s - m - 1, w^i_{k+j} = w^i_{m+j} \).

We now show that the path thus constructed will be admissible for circuit \( \langle F^i, p^i \rangle \), i.e., that, for all \( j = 0, 1, \ldots, s - m - 1, w^i_{k+j} \subseteq W^i(p^i_{r+j}) \).

Since, for all variables of \( Z \) except \( z_i \), the corresponding equations in systems \( F \) and \( F^i \) coincide, then, if such a variable is excited in state \( p_{m+j} \), it will also be excited in state \( p^i_{r+j} \). We consider variable \( z_i \) separately.

Assume that \( z_i \in W_{m+j} \) \((j = 0, 1, \ldots, s - m - 1)\).

1. \( z_i = 1 \). Since, in this case, \( g_i(p_{m+j}) = f_i(p_{m+j}) = 0 \), then \( j = 0 \) and \( z_i \in W^i(p^i_k) \).

2. \( z_i = 0, j > 0 \). Since, in state \( p^i_{m+j} \), \( z_i = 1 \), then, on the basis of property d of segment \( p^i_{r_1} \ldots p^i_{r_s} \) \( W^i(p^i_{m+j}) \subseteq W^i(p^i_{r_s}) \) and, consequently, \( z_i \in W^i(p^i_{r_s}) \).

Thus, the constructed path \( p^i_1 \ldots p^i_k \ldots p^i_{s} \) is admissible for circuit \( \langle F^i, p^i \rangle \), while in state \( p^i_k \) the value of variable \( z_i \) equals 0 and, consequently, \( z_i \in W(p^i_k) \) at the same time as \( z_i \notin W(p_s) \).

This completes the proof of necessity.
Sufficiency. We now show that, if the circuits \(< F, p_i >\) and \(< F^i, p_i^i >\) are not strictly equivalent, then, from this, there follows the existence of a path, admissible for circuit \(< F, p_i >\), with a segment \(p_ip_{i+1} \ldots p_s\), possessing properties a-d.

Circuit \(< F^i, p_i^i >\) will not be strictly equivalent to circuit \(< F, p_i >\) if and only if there exists a path, admissible for \(< F^i, p_i^i >\), to which there does not correspond any path which is admissible for \(< F, p_i >\).

Let \(i = p_ip_{i+1} \ldots p_{k+1}\) be the minimal of those paths which are admissible for circuit \(< F_i, p_i >\). Discarding its last state, we obtain path \(p_ip_{i+1} \ldots p_k\) to which there corresponds path \(l = p_lp_{l+1} \ldots p_s\), which is admissible for circuit \(< F, p_i >\).

As has already been mentioned, the path of vector \(Z\) corresponding to path \(i\) will not be admissible for circuit \(< F, p_i >\) only in the case when \(z_1 \in \mathbb{W}^i(p_i^i)\) and \(z_1 \notin \mathbb{W}(p_s)\).

We now consider what value variable \(z_i\) can have in state \(p_s\).

Assume that \(z_i = 0\). Since, variable \(z_i\) is excited in state \(p_i^i\), while in state \(p_s\) it is not, then \(g_i(p_s) = f_i(p_s) = 0\) and the value of variable \(z_i\) in state \(p_i^i\) equals 1.

Let \(p_k^i\) be the state of path \(i\) beginning with which the value of \(z_i\) became equal to 0, thereafter not changing on the segment of the path until \(p_k^i\).

1. \(p_k^i = p_i\). On the entire segment \(p_ip_{i+1} \ldots p_s\) and, consequently, also on segment \(p_{i+1} \ldots p_k\), the value of functions \(g_i\) and \(f_i\) equals zero. (Otherwise, variable \(z_i\) would have been excited in some state of path \(p_ip_{i+1} \ldots p_s\), and, by virtue of the semimodularity of circuit \(< F, p_i >\), would also have been excited in state \(p_s\).)

Since \(f_i(p_i^i) = 0\), the value of \(z_i\) in this state also equals 0. In the present case, this value cannot be changed on segment \(p_{i+1} \ldots p_k\), which contradicts the initial hypothesis.

2. \(p_k^i = p_i\). In state \(p_{i+1} \ldots p_k\), \(z_i = 0\) and \(f_i(p_{i+1}) = 0\). Consequently, on the path segment \(p_i^i \ldots p_k\) there exists a state \(p_k^i\) such that \(f_i(p_k^i) = 1\). But then, in the state \(l\) of the path equivalent to \(p_k^i\), variable \(z_i\) will be excited and, by virtue of the semimodularity of circuit \(< F, p_i >\), it will also be excited in state \(p_s\), which contradicts the original hypothesis. Hence, it follows that variable \(z_i\) cannot have a value equal to 0 in state \(p_s\).

Thus, \(z_i = 1\) in the states \(p_s\) and \(p_k^i\) and, consequently, \(g_i(p_s) = 0\) and \(f_i(p_s) = 1\), while the value of variable \(z_i\) in state \(p_k^i\) equals 0.

We now consider the two cases corresponding to the possible values of variable \(z_i\) in state \(p_i^i\).

1. \(z_i = 0\). Then, since variable \(z_i\) is not excited in state \(p_i^i\), \(f_i(p_i^i) = f_i(p_i) = 0\).
2. \(z_i = 1\). Since, \(z_i = 0\) in state \(p_i^i\), on segment \(p_{i+1} \ldots p_k\) and, consequently, also on segment \(p_ip_{i+1} \ldots p_s\), there exists a state in which the value of function \(f_i\) equals 0.

Now let \(p_s\) be the closest state to \(p_s\) such that \(f_i(p_s) = 0\) and \(f_i(p_{s+1}) = 1\). On segment \(p_{s+1} \ldots p_s\), we consider an arbitrary state \(p_{s+1}\) in which \(z_i = 1\). We now show that \(f_i(p_{s+1}) = 1\).

Two cases are possible.

1. \(f_i(p_{s+1}) = 0\); i.e., \(r - 1 = l\).

1a. The value of variable \(z_i\) in state \(p_{s+1}\) equals 1. If \(g_i(p_{s+1}) = 0\), then variable \(z_i\) is excited in state \(p_{s+1}\), but, since \(z_i\) has one and the same value in states \(p_r\) and \(p_{s+1}\), by virtue of the semimodularity of circuit \(< F, p_i >\) it must also be excited in state \(p_r\). On segment \(p_r \ldots p_s\), \(z_i\) cannot change its value; consequently, it will also be excited in state \(p_s\), which contradicts the initial hypothesis. Thus, \(g_i(p_{s+1}) = 1\).

1b. The value of variable \(z_i\) equals 0 in state \(p_{s+1}\). Then \(z_i\) must be excited in state \(p_{s+1}\) and, consequently, \(g_i(p_{s+1}) = 1\).

2. \(f_i(p_{s+1}) = 1\); i.e., \(r - 1 > l\).

Let \(p_{s+1}^i\) be a state of path \(i\) equivalent to state \(p_{s+1}\) with respect to \(Z\). It is obvious that the value of variable \(z_i\) cannot equal 1 in this and in all the other states on the path segment \(p_{i+1}^i \ldots p_k^i\) (otherwise, \(z_i\) would retain this value in state \(p_k^i\) inclusive).

2a. The value of variable \(z_i\) in state \(p_{s+1}\) equals 1. If \(g_i(p_{s+1}) = 0\), then \(z_i\) is excited in state \(p_{s+1}\) but not in state \(p_{s+1}\) and, consequently, the selected path is not minimal, which contradicts the original hypothesis.

2b. The value of variable \(z_i\) equals 0 in state \(p_{s+1}\). Since the value of variable \(z_i\) is one and the same in all the states of the path segment \(p_{i+1}^i \ldots p_k^i\), state \(p_{s+1}^i\) is equivalent to state \(p_{s+1}\). Therefore, the value of \(z_i\)
in state $p_{k+1}$ equals 1, whence $z_{i} \in W_{i}(p_{k})$ and $g_{i}(p_{k}) = g_{i}(p_{k+1}) = 1$.

We have thus shown that segment $p_{k}p_{k+1}$ of path $l$ possesses all four properties a-d. This completes the proof of Theorem 2.

In the following theorem sufficient conditions are formulated for a simple disjunctive decomposition of a semimodular circuit to also be semimodular.

In this theorem and in the analogous theorem proposed for simple conjunctive decomposition, variable $z_{i}$ in state $p_{i}$, by which decomposition is accomplished, is not excited.

**THEOREM 3.** Let functions $f_{i}$ and $g_{i}$ be such that there does not exist any path which is admissible for circuit $\langle F, p_{i} \rangle$ along which either $f_{i}$ or $g_{i}$ changes its value twice. Then, if circuit $\langle F^{1}, p_{i}^{1} \rangle$ is strictly equivalent to circuit $\langle F, p_{i} \rangle$ with respect to $Z$, it is semimodular.

**Proof.** Let $p_{k}^{1}$ and $p_{k+1}^{1}$ be two contiguous states in a path which is admissible for circuit $\langle F^{1}, p_{i}^{1} \rangle$, where in variable $z$ has one and the same value in them and is excited in state $p_{k}^{1}$. We now show that it is also excited in state $p_{k+1}^{1}$.

Let $p_{s}$ and $p_{s+1}$ be states of circuit $\langle F, p_{i} \rangle$ which are equivalent to states $p_{k}^{1}$ and $p_{k+1}^{1}$ with respect to $Z$. We carry out the proof that circuit $\langle F^{1}, p_{i}^{1} \rangle$ is semimodular with respect to any variable $z \in Z_{i}$ in three steps.

1. $z = z_{i}, z_{i}$. On the basis of Lemma 2, variable $z$ will be excited in state $p_{s}$ and, by virtue of the semimodularity of circuit $\langle F, p_{i} \rangle$, it will also be excited in state $p_{s+1}$. Since the equations for variable $z$ in systems $F$ and $F^{1}$ coincide, it will also be excited in state $p_{k+1}$.

2. $z = z_{i}$.

2a. The value of variable $z_{i}$ in states $p_{k}^{1}$ and $p_{s}$ equals 1. Then $g_{i}(p_{k}^{1}) = f_{i}(p_{k}^{1}) = 0$, and the value of $z_{i}$ in state $p_{s}^{1}$ equals 0. It is obvious that the value of $z_{i}$ remains equal to 0 in state $p_{k+1}^{1}$. By virtue of the semimodularity of circuit $\langle F, p_{i} \rangle$, $z_{i}$ is also excited in state $p_{s+1}^{1}$. Consequently, $g_{i}(p_{s+1}) = f_{i}(p_{s+1}) = f_{i}(p_{k+1}) = 0$. Since $f_{i}(p_{k+1}) = 0$ and the value of $z_{i}$ in state $p_{k+1}^{1}$ equals 0, variable $z_{i}$ will be excited in this state.

2b. The value of variable $z_{i}$ equals 0 in states $p_{k}^{1}$ and $p_{s}$.

We assume the contrary: $z_{i}$ is excited in states $p_{k}^{1}$, $p_{s}$, and $p_{s+1}$, but is not excited in state $p_{k+1}^{1}$. Then $g_{i}(p_{k+1}) = g_{i}(p_{s+1}) = 0$, $f_{i}(p_{s+1}) = 1$, and the value of variable $z_{i}$ in state $p_{k+1}^{1}$ equals 0. We assume that $g_{i}(p_{s}) = g_{i}(p_{k}) = 0$. Then the value of variable $z_{i}$ in state $p_{k+1}^{1}$ equals 1. Since the value of $z_{i}$ in state $p_{k+1}^{1}$ equals 0, variable $z_{i}$ will be excited in this state.

3. $z = z_{i}$. We assume the contrary: $z_{i}$ is excited in states $p_{k}^{1}$, $p_{s}$, and $p_{s+1}$.

3a. The value of variable $z_{i}$ equals 0 in states $p_{k}^{1}$ and $p_{k+1}^{1}$, while $f_{i}(p_{k}) = 1$ and $f_{i}(p_{k+1}) = 0$. We now show that, on path segment $p_{k}^{1} \ldots p_{k}$, there exists a state $p_{r}$ for which $f_{i}(p_{r}) = 0$.

If $f_{i}(p_{r}) = 0$, then $p_{r} = p_{k}^{1}$.

If $f_{i}(p_{r}) = 1$, then the value of $z_{i}$ in state $p_{r}$ also equals 1. Since $z_{i} = 0$ in state $p_{s}$, on segment $p_{r} \ldots p_{s}$, there necessarily exists a state $p_{r}$ for which $g_{i}(p_{r}) = f_{i}(p_{r}) = 0$.

3b. The value of variable $z_{i}$ equals 1 in state $p_{k}^{1}$, while $f_{i}(p_{k}) = 0$ and $f_{i}(p_{k+1}) = 1$. The proof of this case is analogous to the proof of case 3a.

This completes the proof of Theorem 3.
It should be mentioned that, in the proof of step 1 and of the first part of step 2, no constraints were placed on the character of the changes of functions $g_{i_1}$ and $f_{i_1}$.

We now consider an example of a simple disjunctive decomposition. Let circuit $\langle F, p_1 \rangle$ be specified by the system of equations
\[
\begin{align*}
x_1 &= 1, \\
x_2 &= 1, \\
a &= a\overline{x}_2 \lor \overline{x}_1 x_2, \\
b &= x_2 \lor x_1 a
\end{align*}
\]
and let the initial state be $p_1 = 0010$ (the variables of the circuit are ordered in accordance with the ordinal numbers of the equations, i.e., in state $p_1$, $x_1 = 0$, $x_2 = 0$, $a = 1$, and $b = 0$). The excitation diagram for this circuit is shown in Fig. 1.* As is obvious from the diagrams, circuit $\langle F, p_1 \rangle$ is semimodular and is risk-free in all variables.

Now, as the result of the decomposition of circuit $\langle F, p_1 \rangle$ by variable $a$, let there have been obtained the circuit $\langle F_a, p_2 \rangle$, defined by the system of equations
\[
\begin{align*}
x_1 &= 1, \\
x_2 &= 1, \\
a &= a\overline{x}_2 \lor a_1, \\
a_1 &= x_1 x_2, \\
b &= x_2 \lor x_1 a
\end{align*}
\]
and initial state $p_2 = 00100$. This decomposition corresponds to $g_{a_1} = a\overline{x}_2$, $f_{a_1} = x_1 x_2$.

Path $0010, 0110$ is admissible for circuit $\langle F, p_1 \rangle$ and possesses all four properties of Theorem 2; thus, circuit $\langle F_a, p_2 \rangle$ will not be strictly equivalent to the original circuit.

The excitation diagram of circuit $\langle F_a, p_2 \rangle$ is shown in Fig. 2. It is clear that this circuit has dynamic risk by variable $a$, which means that it is not even equivalent to the original circuit.

If, for the decomposition of the same equation, we set $g_{a_1} = a\overline{x}_2 \lor x_1 a$, $f_{a_1} = x_1 x_2$, then no segment admissible for a path of circuit $\langle F, p_1 \rangle$ will possess all four properties of Theorem 2. Therefore, the decomposition thus obtained will be strictly equivalent to the original circuit. The excitation diagram for this decomposition is shown in Fig. 3.

Since for circuit $\langle F, p_1 \rangle$ there exists an admissible path along which function $f_{a_1}$ changes its value twice, this decomposition can be nonsemimodular. Indeed, in states $01100$ and $11100$ (and also in states $01101$ and $11101$) the conditions for semimodularity by variable $a_1$ are not met.

**Simple Conjunctive Decomposition**

Let $\langle F^1, p_1^1 \rangle$ be a simple conjunctive decomposition of semimodular circuit $\langle F, p_1 \rangle$.

**THEOREM 4.** Circuit $\langle F^1, p_1^1 \rangle$ is strictly equivalent to circuit $\langle F, p_1 \rangle$ relative to set $Z$ of variables if and only if no path which is admissible for circuit $\langle F, p_1 \rangle$ contains a segment $p_t p_{t+1} \ldots p_s$ with the following properties:

a) The value of variable $z_1$ in state $p_s$ equals 0;

*In Figs. 1-3, for the variables $x_1$ and $x_2$, we have provided only those arrows which are essential for the case being considered.*
b) \( f_i(p_f) = 1, \ f_i(p_s) = 0 \);

c) \( g_i(P_s) = 1 \);

d) for each \( r \) (\( l < r < s \)) it follows from the fact that the value of \( z_i \) in state \( p_r \) equals 0, while \( f_i(p_r) = 0 \), that \( g_i(p_{r-1}) = 0 \).

The formulation of the theorem on semimodularity for simple conjunctive decomposition coincides exactly with the formulation of Theorem 3.

The proofs of these theorems are analogous to the proofs of the corresponding theorems for simple disjunctive decomposition and, therefore, will not be adduced here.

**Inverse Decomposition**

**Theorem 5.** The inverse decomposition \( \langle F^i, p^i \rangle \) of semimodal circuit \( \langle F, p_i \rangle \) is strictly equivalent to this circuit relative to set \( Z \) of variables.

**Proof.** We assume the contrary: There exists a path, admissible for circuit \( \langle F^i, p^i \rangle \), to which there does not correspond any path admissible for circuit \( \langle F, p_i \rangle \). Let \( l^i = p^i_{1}p^i_{2}...p^i_{k+1} \) be the minimal such path, and let \( l = p_{1}...p_{s} \) be a path which is admissible for circuit \( \langle F, p_i \rangle \) and which corresponds to segment \( p^i_{1}...p^i_{k} \). Then, as was shown earlier, \( z_i \in W(p^i_k) \) and \( z_i \notin W(p_s) \).

1. We assume that the value of variable \( z_i \) equals 0 in states \( p^i_k \) and \( p_s \). Then \( f_i(p_s) = 0 \), and the value of variable \( z_i \) equals 0 in state \( p_s \).

Let \( p^i_{1} \) be the state of path \( l^i \), beginning with which the value of variable \( z_i \) in all states of this path through \( p^i_k \) equals 0.

1a. \( p^i_{1} \neq p^i_{i} \). Then the value of variable \( z_i \) equals 1 in state \( p^i_{1} \), and \( f_i(p^i_{1}) = 0 \). Consequently, on a path segment \( p^i_{1}...p^i_{k} \) there exists a state \( p^i_{j} \) for which \( f_i(p^i_{j}) = 1 \). But then, in the state of path \( l \) which is equivalent to it, variable \( z_i \) will be excited and, by virtue of the semimodularity of circuit \( \langle F, p_i \rangle \), it will also be excited in state \( p_s \), which contradicts the initial hypothesis concerning paths \( l^i \) and \( l \).

1b. \( p^i_{1} = p^i_{j} \). Here \( f_i(p^i_{1}) = 0 \), otherwise \( z_i \) would be excited in state \( p^i_{1} \) and, by virtue of the semimodularity of circuit \( \langle F, p_i \rangle \), it would also be excited in state \( p_s \).

Since variable \( z_i \) is not excited in state \( p^i_{1} \), its value equals 1. This value will be retained on all the segments of path \( p^i_{1}...p^i_{k} \), since function \( f_i \) will not change its value on this segment. Thus, variable \( z_i \) must have the value 1 in state \( p^i_k \).

Thus, in both cases, we arrive at a contradiction.

2. If we assume that the value of variable \( z_i \) equals 1 in states \( p^i_k \) and \( p_s \), then, completely analogous to our earlier argument, we also arrive at a contradiction. This completes the proof of the theorem.

**Theorem 6.** Inverse decomposition \( \langle F^i, p^i \rangle \) of semimodular circuit \( \langle F, p_i \rangle \) is semimodular.

**Proof.** Let \( p^i_k \) and \( p^i_{k+1} \) be two neighboring states in a path which is admissible for circuit \( \langle F^i, p^i \rangle \), where variable \( z \) has the same value in both of them and is excited in state \( p^i_k \).

We now show that it will also be excited in state \( p^i_{k+1} \).

Let \( p^i_s \) and \( p^i_{s+1} \) be the states of circuit \( \langle F, p_i \rangle \) which are equivalent to states \( p^i_k \) and \( p^i_{k+1} \). We perform the proof this theorem, just as that for Theorem 3, in three steps.

1. \( z = z_i \), \( z_i \). The proof coincides with the proof of the corresponding step in Theorem 3.

Before we turn to the proof of the following steps, we shall formulate an assertion of whose validity it is easy to convince oneself.

**Assertion 2.** In any state belonging to a path which is admissible in circuit \( \langle F^i, p^i \rangle \), the variables \( z_i \) and \( z_i \) cannot be excited simultaneously.

2. \( z = z_i \). As follows from Assertion 2, variable \( z_i \) is not excited in state \( p^i_k \) and, consequently, it has the same value in state \( p^i_{k+1} \) as in state \( p^i_k \). It immediately follows from this that variable \( z_i \) will also be excited in state \( p^i_{k+1} \).
3. \( z = z_i \). We note, first of all, that states \( p_s \) and \( p_{s+1} \) are also contiguous states in some path which is admissible for circuit \( \langle F, p_1 \rangle \).

We assume the contrary: \( z_i \in W^i(p_k) \) and \( z_i \notin W^i(p_{k+1}) \). Then \( f_1(p_k) \neq f_1(p_{k+1}) \) and, consequently,

\[
\tilde{f}(p_s) \neq \tilde{f}(p_{s+1}).
\]

Moreover, by virtue of Assertion 2, variable \( z_i \) is not excited in state \( p_k \), so that \( p_s \sim p_{s+1} \), and, consequently,

\[
p_s \sim p_{s+1}.
\]

After having compared the values of variables \( z_i \) and \( z_i \) and of function \( f_1 \) in state \( p_k \), we can convince ourselves that the value of variable \( z_i \) does not coincide with the value of \( f_1(p_k) \). Obviously, this is also the case in state \( p_s \). Thus, in state \( p_s \), variable \( z_i \) will be excited. It follows from this, and from (4) and (5), that variable \( z_i \) is not excited in state \( p_{s+1} \), which contradicts the condition of semimodularity of circuit \( \langle F, p_1 \rangle \).

This completes the proof of Theorem 6.

In this paper we have used the following notation: \( S \) is an asynchronous logical circuit; \( Z \) is the set of variables of circuit \( S \); \( F \) is the system of equations of circuit \( S \) which specifies this circuit; \( E \) is the set of all states of circuit \( S \); \( \varphi : E \to E \) is the mapping induced by the system of equations of \( F \); \( I(p_1, p_2) \) is the minimal interval containing states \( p_1 \) and \( p_2 \); \( W(p) \) is the set of variables which are excited in state \( p \); \( W_k \) is the set of all variables whose values differ as between states \( p_k \) and \( p_{k+1} \) of the path under consideration; \( F^z \) and \( F^i \) are decompositions of circuit \( F \) by the \( i \)-th variable; \( \pi Z {p} \) is the projection of state \( p \) on set of variables \( Z \); \( p_1 \sim p_2 \) is an equivalence relation of states \( p_1 \) and \( p_2 \) relative to set of variables \( Z \);

\[
\pi_z p_1 = \pi_z p_2.
\]

**LITERATURE CITED**


**ALGEBRAIC TREATMENT OF DATA STRUCTURES**

V. V. Bublik and S. S. Gorokhovskii

UDC 519.766.2

Algebraic methods of investigation are gaining ever wider acceptance in the theory and practice of programming. The importance of such methods has been emphasized in [1].

With regard to the theory of data structures, the algebraic approach makes it possible to exactly formulate the principal concepts of the theory which are fully adequate for what intuition suggests. This approach serves as a basis for the elaboration of the algorithmic language L2 [2, 3]. An important step in the construction of a general theory of data structures has been made in [4]. In it, a concept of data structures that incorporates in a natural manner both homogeneous data structures (such as arrays, registers, rows of symbols and bits, etc.) and inhomogeneous structures (such as strings, the structures of the languages PL/1 and ALGOL-68, trees, graphs, networks, multiconnected hierarchical structures, etc.) has been introduced. The main attention in [4] has been devoted to homogeneous data structures.

Let us note the following important aspect of the theory of data structures. Whereas conventional objects (such as trees, strings, and rows) admit a simple and natural algebraic treatment, the more complex and highly developed structures (such as composite objects, networks, and multiconnected structures) require the use of the concepts of the theory of graphs in the algebraic treatment; moreover, this is not always done just for greater clarity. (For example, in [2], the operations over composite objects are defined with the aid of vertices and arcs.) The use of graph-theoretical concepts can apparently be explained by the absence of a general algebraic procedure of representation of composite objects that could be used for expressing the structure of (elementary and composite) data, and also by the recursive character of the data.