# Two inductively coupled RLC circuits

Two inductively coupled RLC circuits are shown in Figure 1. Having 2 circuits gives 2 resonant frequencies whose separation depends on the value of the *mutual inductance* M (the ratio of the voltage in the secondary to the rate of change of primary current with time, and the unit is the henry. This has a reactance at the operating frequency $X_m = \omega M$).
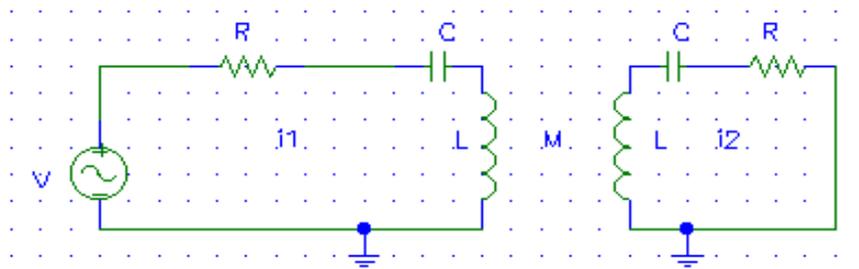
The mutual inductance coupling between primary and secondary can be related to their self-inductance by means of the coupling constant k:

$$k = \frac{M}{\sqrt{L_1 \cdot L_2}} \tag{1}$$

Notice, that since k is defining the relationship between magnetic flux linkages in the circuit, it can never be greater than 1. A value of 1 means that all the flux produced by the primary is linked with the secondary and vice versa. A value of k greater than 1 would mean that more than all of the flux produced by the primary is linked with the secondary.

The coupling constant is independent of the number of turns in a coil. The number of turns in a coil determines the magnetic field, which will be produced for a given current. The coupling constant is concerned with how the lines of magnetic force produced by one coil interact with another coil, and hence the coupling constant between two air spaced coils depends only on their physical size and disposition in space. Hence to obtain the best coupling between primary and secondary in an air-cored transformer we can only change the size and spatial relationships of the coils.

Kirchhoff's voltage law equations for the primary and secondary loops are given by

$$\left[R + j\omega L + \frac{1}{j\omega C}\right] \cdot I_1 + j\omega M \cdot I_2 = V_0$$

(2)

$$j\omega M \cdot I_1 + \left[R + j\omega L + \frac{1}{j\omega C}\right] \cdot I_2 = 0$$

(it is assumed that $R_1 + R_{L1} = R_2 + R_{L2} = R$ and $L_1 = L_2$).

We can write these eqs. in the matrix form as follows

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_0 \\ 0 \end{bmatrix},$$

(3)

where, $a = \left[R + j\omega L + \frac{1}{j\omega C}\right]$ and $b = j\omega M$.

Following Cramer's rule $I_1 = \frac{\Delta_1}{\Delta}$ and $I_2 = \frac{\Delta_2}{\Delta}$,

where $\Delta_1 = \begin{vmatrix} V_0 & b \\ 0 & a \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a & V_0 \\ b & 0 \end{vmatrix}$, and $\Delta = a^2 - b^2$.

Thus, the solution from which the frequency response can be obtained is:

$$I_1 = \frac{aV_0}{a^2 - b^2} \text{ and } I_2 = -\frac{bV_0}{a^2 - b^2}.$$

(4)

Resonance occurs at the 2 frequencies given by the following equations:

$$\omega_1^2 = \frac{1}{LC \cdot \left[1 - \dfrac{M}{L}\right]} \text{ and } \omega_2^2 = \frac{1}{LC \cdot \left[1 + \dfrac{M}{L}\right]}$$

(5)

Here, the coupling coefficient $k = M/L$ (for $L_1 = L_2$).

The behavior of the circuit can be understood qualitatively on the basis of the *reflected impedance* (or *coupled impedance*).

A transformer (or inductively coupled circuit) is said to "reflect" impedance in the secondary into the primary circuit. Consider the coupled circuits shown in Fig 2. The positive direction of the currents is

chosen into the polarity mark on the generator representing the induced voltages, so that Kirchhoff's equations are

$$\mathbf{Z_1I_1} + \mathbf{Z_MI_2} = \mathbf{V_1}, \text{ and } \mathbf{Z_MI_1} + \mathbf{Z_2I_2} = 0. \text{ (see Eq.2)}$$

Z_M is the mutual impedance jωM, Z_1 includes the source impedance, and Z_2 the secondary load. These equations may be solved for the equivalent primary impedance

$$\mathbf{Z_e} = \mathbf{V_1/I_1} = \mathbf{Z_1} - \mathbf{Z_M}^2/\mathbf{Z_2}. \text{ (from eqs. (2) and (4))}$$

The reflected impedance is then ω^2M^2/Z_2. Note that a resistance is reflected as a resistance, a capacitance as an inductance ($\mathbf{Z_e} \approx \mathbf{Z_1} + \omega^2M^2 \cdot j\omega C$), and an inductance as a capacitance ($\mathbf{Z_e} \approx \mathbf{Z_1} + \omega^2M^2/ j\omega L$).
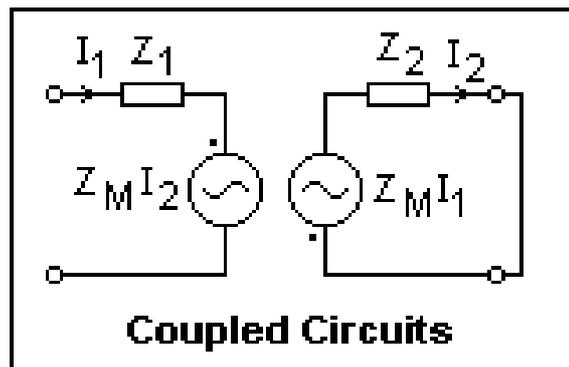

Fig.2.

At resonance, the reflected impedance is resistive, and acts to lower the Q of the primary, and thereby to reduce the output. This is counteracted by the increased coupling, which increases the output. The lower Q gives a wider passband. At frequencies lower than exact resonance, the reflected impedance is inductive, which adds to the inductance of the primary and resonates at a lower frequency, producing a peak in the output. At frequencies higher than exact resonance, the reflected impedance is capacitive, which cancels part of the inductance and causes the circuit to resonate at a higher frequency, producing the other peak.

As the coupling is reduced, the response becomes single-peaked at *critical coupling*, and then decreases as the coupling is made even looser.

The critical coupling coefficient is given by $k_c = \dfrac{1}{\sqrt{Q_1 \cdot Q_2}}$,

in terms of the Q's of the individual tuned circuits.

Frequently it is assumed that *optimum* coupling occurs for k ≈ 1.5k$_c$. In this case, the response is double-peaked, but does not dip much between the peaks and the response is close to what is ideally required. The bandwidth can be estimated as BW = kf$_o$, where k is the coefficient of coupling and f$_o$ is the resonant frequency of each circuit.

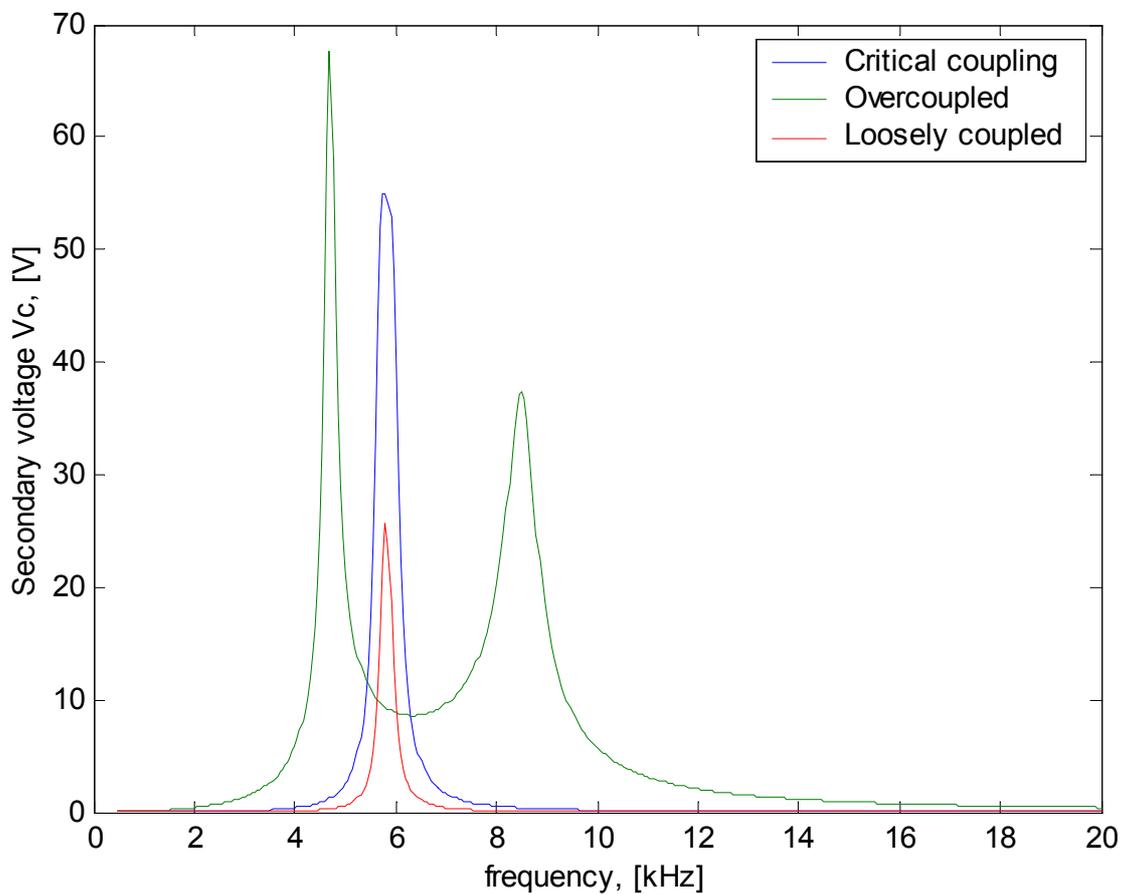Figure 3 shows an example of secondary circuit response *Vc(f)* (voltage measured on capacitor) for different coupling coefficients.

Fig. 3.

(R = 250 Ohm, C = 5 nF, L = 0.15 H, Vin =5V)