5. Mutual Inductance

To develop an equivalent coupling network for an air-core transformer, we require one parameter which is common to both primary and secondary circuits. This parameter is the **mutual inductance** of the pair of coils. In Chapter 9, we found that a coil has a *self*-inductance of one henry when current in that coil *changing* at the rate of one ampere per second induces a cemf of one volt in the coil. Applying the same line of thought to the process of mutual induction, we can say that

> **A pair of magnetically-coupled coils has a mutual inductance of one henry when current changing at the rate of one ampere per second in one coil induces an average emf of one volt in the other coil.**
>
> **The letter symbol for mutual inductance is M.**

Therefore, by definition, $$E_{s_{av}} = \frac{MI_{p_m}}{t} \tag{19.28}$$

Since the instantaneous current must rise from zero to maximum in one quarter of a cycle of a sine wave,

$$E_{s_{av}} = \frac{MI_{p_m}}{1/4f} = 4fMI_{p_m}$$

But $$E_{av} = \frac{2}{\pi}E_m \tag{11.15}$$

Therefore $$\frac{2}{\pi}E_{s_m} = 4fMI_{p_m}$$

from which $$E_s = 2\pi fMI_p \tag{19.29}$$

Rearranging Equation (19.29) gives

$$\frac{E_s}{I_p} = 2\pi fM = X_M \tag{19.30}$$

where X_M is called the **mutual reactance** *of the magnetically coupled windings.*

Example 6: 120 v, 60 c is applied to the primary of a transformer whose primary inductance is 5 h. The open-circuit secondary voltage is 40 v. Neglecting losses, what is the mutual inductance between the two windings?

Solution:

$$E_s = \omega MI_p$$

But $$I_p = \frac{E_p}{\omega L_p}$$

$$\therefore \quad E_s = \frac{M}{L_p}E_p$$

And $$M = L_p\frac{E_s}{E_p} = 5 \times \frac{40}{120} = \mathbf{1.67\ h}$$

Writing Equation (19.28) in terms of the instantaneous emf induced into the secondary,

$$e_s = M\frac{di_p}{dt} \tag{19.28}$$

And since we are concerned only with sine-wave primary currents,

$$i_p = I_m \sin \omega t \quad \text{and} \quad \frac{di_p}{dt} = \omega I_m \cos \omega t$$

Hence, $$e_s = \omega MI_m \cos \omega t \tag{19.31}$$

Since the instantaneous emf in Equation (19.31) will be at its peak value when $\cos \omega t = 1$,

$$E_m = 2\pi fMI_m$$

from which $$X_m = \frac{E_{m(\text{sec})}}{I_{m(\text{pri})}} = 2\pi fM \tag{19.30}$$

Combining Equations (18.1) and (19.28),

$$\frac{N\Phi_m}{t} = \frac{MI_{p_m}}{t}$$

But from Equation (19.27), $$\Phi_m = K\Phi_p$$

$$\therefore \quad MI_{p_m} = N_sK\Phi_p \tag{1}$$

If we now reverse the windings and use the original secondary winding as the primary and vice versa, it follows that

$$MI_{s_m} = N_pK\Phi_s \tag{2}$$

Multiplying Equations (1) and (2),

$$M^2 = K^2\left(\frac{N_p\Phi_p}{I_{p_m}}\right)\left(\frac{N_s\Phi_s}{I_{s_m}}\right) \tag{3}$$

But from the definition of self-inductance,

$$E_{av} = \frac{LI_m}{t}$$

and from the definition of the weber,

$$E_{av} = \frac{N\Phi}{t}$$

$$\therefore \quad L = \frac{N\Phi}{I_m}$$

Substituting in Equation (3) gives

$$M = K\sqrt{L_pL_s} \tag{19.32}$$

Example 7: If the secondary winding of the transformer of Example 6 has a self-inductance of 0.8 h, what is the coefficient of coupling between the windings?

Solution:

$$K = \frac{M}{\sqrt{L_pL_s}} = \frac{1.67}{\sqrt{5 \times 0.8}} = \mathbf{0.83}$$

One method for experimentally determining mutual inductance is shown in Fig. 19.11. We can measure the total inductance of the series-connected coils on an inductance bridge. Connecting the windings as shown and checking by our hand rule, the flux produced by the current in one winding is in the same direction around the core as the flux produced by the current in the other winding. This increases the total flux, thus increasing the emf induced by a given alternating current; thus increasing the total inductance. All the induced emf's will be in phase. There are four induced emf's: the self-induced emf in the primary, the emf mutually induced in the primary by current in the secondary, the emf mutually induced in the secondary by current in the primary and the self induced emf in the secondary.

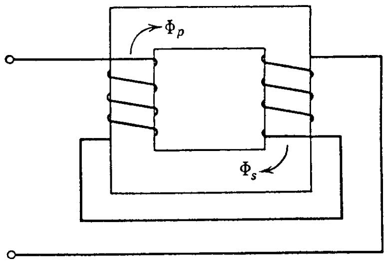

Fig. 19.11. Determining mutual inductance.

$$\therefore \quad E = I(\omega L_p + \omega M + \omega M + \omega L_s)$$

from which $L_T = L_p + L_s + 2M$.

If, however, we reverse the leads to the secondary, the mutually induced emf's in each coil are 180° out of phase with the self-induced emf's resulting in

$$L_T = L_p + L_s - 2M$$

Therefore, we can extend our original Equation (9.6) for two inductances in series to include magnetic coupling between them.

$$\therefore \quad L_T = L_p + L_s \pm 2M \tag{19.33}$$

The mutual inductance of the magnetically coupled coils in Fig. 19.11 will be one-quarter of the difference between the total inductance readings with the coils connected series aiding and then series opposing.

6. Coupled Impedance

Before we set up the equivalent coupling network for a loosely coupled transformer, we should note a useful practical advantage that transformer coupling has over other forms of coupling networks. For a given direction

of $\boldsymbol{E}_1$ in Fig. 19.12, we can reverse the phase of the output voltage by reversing the direction of one of the windings, or simply by reversing the leads to one of the windings. Where it is necessary to keep track of this phase relationship in a circuit diagram, we mark one end of each winding with a dot as shown in Fig. 19.12(a) and (b). If, at a certain moment, changing mutual flux induces an instantaneous emf into the primary winding with a polarity such that the dotted end of the winding is positive with respect to the undotted end, then the same mutual flux change must induce an instantaneous emf into the secondary winding with a polarity such that the dotted end of the secondary winding is positive with respect to the undotted end.

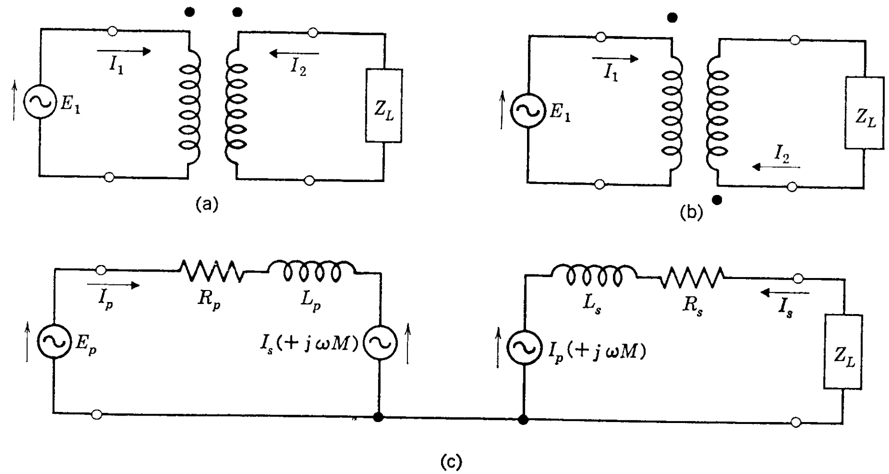

Fig. 19.12. An equivalent circuit of an air-core transformer.

Although reversing the secondary polarity will reverse the current through the external circuit, note that $\boldsymbol{I}_2$ in Fig. 19.12(b) still must flow into the dot end of the secondary winding just as it does in Fig. 19.12(a). Hence, the primary circuit is not affected by any phase reversal we obtain by reversing the secondary leads. In selecting the direction for these current arrows, we must remember that the *same* mutual flux (which is produced by these currents) induces emf's into both primary and secondary windings. So if we draw $\boldsymbol{I}_1$ pointing into the dot end of the primary winding, we must also draw $\boldsymbol{I}_2$ pointing into the dot end of the secondary winding.

When we write the Kirchhoff's voltage law (loop) equation for the primary loop, there are three voltages which add up (vectorially) to equal $\boldsymbol{E}_1$: an IR drop across the resistance of the primary circuit, a *self*-induced emf due to $\boldsymbol{I}_1$ flowing through the primary winding, and a *mutually* induced emf due to $\boldsymbol{I}_2$ flowing in the secondary circuit. Figure 19.12(c) shows one form of equivalent circuit for this transformer, representing the self-induced emf by an inductance symbol and the mutually induced emf by a generator symbol. To satisfy Lenz's law, the polarity of the generator must be such that it opposes changes in $\boldsymbol{I}_1$ due to the applied emf $\boldsymbol{E}_1$. Hence, changing the polarity of the generator symbol in the secondary circuit will *not* change the polarity of the voltage $\boldsymbol{I}_s(+j\omega M)$ in the primary circuit. Consequently, Kirchhoff's voltage law equation for the primary loop becomes

$$\boldsymbol{E}_p = \boldsymbol{I}_p(R_p + j\omega L_p) + \boldsymbol{I}_s(+j\omega M)$$

or

$$\boldsymbol{E}_p = \boldsymbol{I}_p\boldsymbol{Z}_p + \boldsymbol{I}_s(+j\omega M) \tag{19.34}$$

where $\boldsymbol{Z}_p$ is the open-circuit impedance of the primary circuit by itself, and ωM is the mutual reactance of the two windings. Note that Lenz's law gives us the same direction for $\boldsymbol{I}_2$ that we chose for our generalized coupling network. [Compare Equation (19.34) with Equation (19.5).] Hence, we write the loop equation for the secondary circuit of Fig. 19.12(c) as

$$0 = \boldsymbol{I}_s(\boldsymbol{Z}_L + R_s + j\omega L_s) + \boldsymbol{I}_p(+j\omega M)$$

or

$$0 = \boldsymbol{I}_s(\boldsymbol{Z}_L + \boldsymbol{Z}_s) + \boldsymbol{I}_p(+j\omega M) \tag{19.35}$$

where $\boldsymbol{Z}_s$ is the open-circuit impedance of the secondary winding by itself. [Compare Equations (19.35) and (19.14).] From Equation (19.35)

$$\boldsymbol{I}_s = \frac{-\boldsymbol{I}_p(+j\omega M)}{\boldsymbol{Z}_s + \boldsymbol{Z}_L} \tag{19.36}$$

Substituting in Equation (19.34),

$$\boldsymbol{E}_p = \boldsymbol{I}_p\boldsymbol{Z}_p - \boldsymbol{I}_p\frac{(+j\omega M)^2}{\boldsymbol{Z}_s + \boldsymbol{Z}_L} \tag{19.37}$$

Equation (19.37) is the equivalent of Equation (19.16). In this case we can go one step further by substituting -1 for j^2. Thus

$$\boldsymbol{E}_p = \boldsymbol{I}_p\boldsymbol{Z}_p + \boldsymbol{I}_p\frac{(\omega M)^2}{\boldsymbol{Z}_s + \boldsymbol{Z}_L} \tag{19.38}$$

and dividing through by $\boldsymbol{I}_p$,

$$\boldsymbol{Z}_{in} = \boldsymbol{Z}_p + \frac{(\omega M)^2}{\boldsymbol{Z}_s + \boldsymbol{Z}_L} \tag{19.39}$$

where $(\omega M)^2/(\boldsymbol{Z}_s + \boldsymbol{Z}_L)$ is the **coupled impedance** *for transformer coupling.* Note that when we substitute -1 for j^2, $(\omega M)^2$ has now become an impedance with a 0° angle.

We can check Equation (19.39) by considering the effect of loading the secondary winding of a pair of magnetically coupled coils. If the secondary is left open-circuit, $\boldsymbol{Z}_L$ is infinitely large and, therefore, the coupled impedance in Equation (19.39) becomes zero. Therefore, the total impedance is simply the primary impedance alone as we would expect. If we connect a resistance across the secondary winding, $\boldsymbol{Z}_L$ will have a 0° angle. Since $\boldsymbol{Z}_s$ is largely the inductive reactance of the secondary winding, the total secondary circuit impedance is inductive. Therefore, when we carry out the vector division, dividing an impedance with a + angle into $(\omega M)^2$ with its 0° angle results in a *capacitive* coupled impedance. Since $\boldsymbol{Z}_p$ is largely *inductive*, the *capacitive* impedance coupled into the primary circuit via the mutual inductance adds the coupled resistance component in series with the primary resistance component but *reduces* the total primary reactance. As a result, the total impedance becomes smaller as the secondary is loaded with a resistance, thus allowing more primary current to flow in order to transfer energy to the secondary circuit. This then checks with our first approach to transformer action in which we discovered that primary current must increase when a transformer is loaded in order to keep the amplitude of the mutual flux sine wave constant.

Example 8: If the resistance of the windings can be neglected, determine the total input impedance to the transformer of Examples 6 and 7, (a) with the secondary open circuit, and (b) with a 50 ohm resistance connected to the secondary.

Solution:

(a) $$\boldsymbol{Z}_{in} = \boldsymbol{Z}_p = \omega L_p = 377 \times 5 = \boldsymbol{+j1885\ \text{ohms}}$$

(b) $$\text{Coupled } \boldsymbol{Z} = \frac{(\omega M)^2}{\boldsymbol{Z}_s + \boldsymbol{Z}_L} = \frac{(377 \times 1.67)^2}{+j(377 \times 0.8) + 50}$$

$$= \frac{396000\underline{/0^\circ}}{304\underline{/80.5^\circ}} = 1300\underline{/-80.5^\circ} = 213 - j1284 \text{ ohms}$$

$$\therefore \quad \boldsymbol{Z}_{in} = (+j1885) + (213 - j1284) = 213 + j601$$
$$= \boldsymbol{636\ \text{ohms}\underline{/+70.5^\circ}}$$

7. Tuned Transformers

The most important application of loose coupling is the use of tuned transformers in radio circuitry. If we consider the source of emf to be a constant voltage source as in Fig. 19.13, the primary winding is designed to form a series resonant circuit with the capacitive reactance of C_p equal to the inductive reactance of the primary winding at the desired resonant frequency. This will result in maximum current in the primary winding at resonance (neglecting for the moment any coupled impedance due to secondary current).*

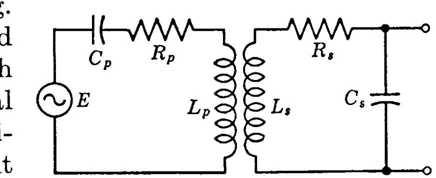

Fig. 19.13. A tuned transformer.

* If a vacuum-tube amplifier is used to feed the primary of a tuned transformer, the vacuum tube represents a constant *current* source providing a signal current equal to the mutual conductance of the tube times the signal input voltage to its grid. Hence C_p must then be connected in *parallel* with the primary winding to obtain maximum primary current (resonant rise of current) at resonance.

Considering now the effect of tuning the secondary circuit, the total secondary circuit impedance [$\mathbf{Z}_s + \mathbf{Z}_L$ in Equation (19.39)] will be minimum and equal to R_s at resonance. Therefore the secondary circuit behaves like a series resonant circuit. Since $(\omega M)^2$ is essentially constant over the small range of frequencies near resonance, when $\mathbf{Z}_s + \mathbf{Z}_L$ becomes a *minimum*, the coupled impedance becomes a *maximum* as shown in Fig. 19.14. And when the source frequency is slightly lower than the resonant frequency, X_C, is greater than X_L, and the secondary impedance becomes capacitive. But dividing a *capacitive* impedance into $(\omega M)^2$ with its 0° angle results in an *inductive* coupled impedance. Therefore, as far as the signal source is concerned, the secondary behaves as if it were a *parallel* resonant circuit in *series* with the primary winding.

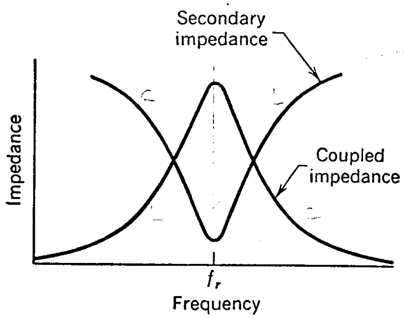

Fig. 19.14. Effect of a tuned secondary circuit.

The primary impedance by itself in the circuit of Fig. 19.13 is attempting to become a minimum at resonance in order to allow maximum primary current to flow. But the coupled impedance is attempting to raise the primary impedance at resonance, thus limiting the maximum primary current. The extent to which the coupled secondary tuned circuit affects the primary resonance curve depends on the degree of coupling between the coils. When the coupling is very loose, the mutual inductance is very small and even at resonance the coupled impedance is *smaller* than the resistance of the primary circuit. Under these circumstances, the only effect that the coupled secondary has on the shape of the primary current resonance curve is to limit its peak value slightly. The secondary current tends to have the usual series resonance curve. But with loose coupling, since the primary current rises to a maximum at resonance and since the secondary induced emf depends on the primary current, the secondary resonance curve is much sharper than that of a single tuned circuit having the same Q.*

We will recall from earlier studies that maximum transfer of energy occurs when the load resistance is equal to the resistance of the source. Applying this to magnetically coupled tuned circuits, maximum energy transfer will take place when the coupled resistance is equal to the primary circuit resistance. Since the coupled resistance depends on the mutual inductance, which in turn depends on the coefficient of coupling, there is a **critical coupling** for a given pair of tuned circuits at which maximum energy transfer from primary to secondary takes place.

At resonance, $$Z_p = R_p$$

and the coupled $$Z = \frac{(\omega M)^2}{R_s}$$

Therefore for critical coupling,

$$M^2 = R_p R_s/\omega^2$$

But $$M^2 = K^2 L_p L_s \tag{19.32}$$

Therefore $$K^2 = \frac{R_p}{\omega L_p} \times \frac{R_s}{\omega L_s}$$

from which $$K_c = \frac{1}{\sqrt{Q_p Q_s}} \tag{19.40}$$

With critical coupling, the secondary current attains its greatest value. But with the coupled impedance rising at resonance, the primary current

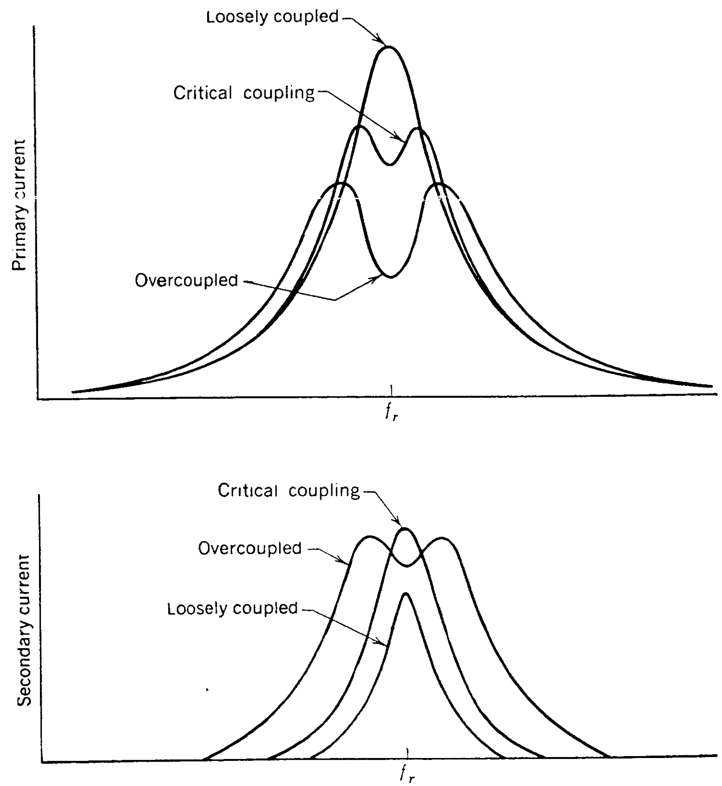

Fig. 19.15. Effect of coefficient of coupling on the resonance curves of a tuned transformer.

* The bandwidth is approximately $\Delta f \approx K f_r$.

is no longer maximum at resonance. As we have already noted, at a frequency slightly below resonance the coupled impedance is *inductive*, whereas the impedance of the series tuned primary is *capacitive*. Therefore there are two frequencies, one slightly above the resonant frequency and one slightly below where the coupled reactance tunes the primary reactance to give minimum total primary impedance and therefore maximum primary current. This double hump in the primary current tends to flatten the peak of the secondary current response curve since the emf induced into the secondary at the resonant frequency is not quite as great as at the two hump frequencies either side of resonance.

If the tuned circuits are overcoupled, the increase in coupled impedance moves the primary current humps further apart. This in turn causes such a decrease in primary current at resonance that the secondary current also starts to show a double humped curve as shown in Fig. 19.15. In practice a coefficient of coupling of 1.5 times the critical coefficient of coupling produces such a slight dip in the secondary current at resonance that the resulting secondary current resonance curve has a very desirable flat top with steep skirts.

Problems

Calculate the open-circuit Z-parameters for each of the following T-networks, using the subscript notation of Fig. 19.3.

1. $\mathbf{Z}_p = R_p = 356$ ohms, $\mathbf{Z}_s = R_s = 36$ ohms, $\mathbf{Z}_m = R_m = 286$ ohms.
2. $\mathbf{Z}_p = R_p = 2500$ ohms, $\mathbf{Z}_s = -j500$ ohms, $\mathbf{Z}_m = R_m = 10$ kilohms.
3. $\mathbf{Z}_p = \mathbf{Z}_s = 20 + j800$ ohms, $\mathbf{Z}_m = -j800$ ohms.
4. $\mathbf{Z}_p = \mathbf{Z}_s =$ a 50 μh inductance in series with a 200 $\mu\mu$f capacitance, $\mathbf{Z}_m =$ a 200 μh inductance, $f = 1$ mc.

Find the equivalent T-network for the "black boxes" with the following Z-parameters.

5. $\mathbf{Z}_{11} = 600$ ohms$\angle 0°$, $\mathbf{Z}_{12} = \mathbf{Z}_{21} = 400$ ohms$\angle 0°$, $\mathbf{Z}_{22} = 1200$ ohms$\angle 0°$.
6. $\mathbf{Z}_{11} = 500\text{ K} - j5\text{ K}$ ohms, $\mathbf{Z}_{12} = \mathbf{Z}_{21} = \mathbf{Z}_{22} = 500\text{ K} + j0$ ohms.
7. $\mathbf{Z}_{11} = \mathbf{Z}_{22} = 50 + j0$ ohms, $\mathbf{Z}_{12} = \mathbf{Z}_{21} = -j1200$ ohms.
8. $\mathbf{Z}_{11} = 100 + j1500$ ohms, $\mathbf{Z}_{12} = \mathbf{Z}_{21} = 40 + j500$ ohms, $\mathbf{Z}_{22} = 80 + j1200$ ohms.
9. What is the coupled impedance when a 1500 ohm resistive load is connected to the output terminals of the network in Problem 5?
10. What is the coupled impedance when a capacitive reactance of 1240 ohms is connected to the output terminals of the network in Problem 8?
11. What is $\mathbf{Z}_{in}$ when a 250 ohm resistive load is connected to the output terminals of the T-network in Problem 1?
12. What is $\mathbf{Z}_{in}$ when a 1 kilohm resistive load is connected to the output terminals of the T-network in Problem 4?
13. In the T-attenuator circuit of Problem 11, determine the ratio between the total power input to terminals (1, 1) and the power into the load at terminals (2, 2) and express the attenuation by the relationship

$$\text{Attenuation (in decibels)} = 10 \log_{10} (P_{in}/P_{load})$$

14. With the techniques used in deriving Equations (19.7) to (19.10), and noting that the internal resistance of the internal generator symbol is zero, find the Z-parameters for the transistor in Fig. 16.23.

Find the open-circuit Z-parameters for each of the following π-networks, using the subscript notation of Fig. 19.6.

15. $\mathbf{Z}_1 = R_1 = 3470$ ohms, $\mathbf{Z}_2 = R_2 = 350$ ohms, $\mathbf{Z}_3 = R_3 = 440$ ohms.
16. $\mathbf{Z}_1 = 80$ kilohms$\angle 0°$, $\mathbf{Z}_2 = 470$ kilohms$\angle 0°$, $\mathbf{Z}_3$ is an 0.02 μf capacitor, and $f = 100$ c. (See Fig. 16.27)
17. $\mathbf{Z}_1 = \mathbf{Z}_2 = -j3600$ ohms, $\mathbf{Z}_3 = +j3600$ ohms.
18. $\mathbf{Z}_1 = \mathbf{Z}_2 = 150$ μh, $\mathbf{Z}_3 = 600$ $\mu\mu$f, $f = 500$ Kc.
19. Find $\mathbf{Z}_{in}$ when a 300 ohm resistive load is connected to the output terminals of the π-network in Problem 15.
20. Find $\mathbf{Z}_{in}$ when a load of $300 - j400$ ohms is connected to the output terminals of the π-network in Problem 18.
21. Hybrid parameters are not used for simple resistance coupling networks. However they may readily be derived. What would the h-parameters for the network of Problem 1 look like?
22. Find the hybrid parameters for the transistor in Fig. 16.23.
23. For a common-base configuration, a certain transistor has the following parameters:

$h_{11} = 40$ ohms $\qquad h_{12} = 4 \times 10^{-4}$
$h_{21} = -0.96 \qquad h_{22} = 10^{-5}$ mho

The transistor feeds into a 4.7 kilohm resistive load and is fed from a 1 kc signal source having an open-circuit emf of 10 mv and an internal resistance of 250 ohms. Calculate the voltage gain (the ratio of output to input voltage).

24. For a common-emitter configuration, the same transistor as used in Problem 23 has the following parameters:

$h_{11} = 1000$ ohms $\qquad h_{12} = 9.6 \times 10^{-3}$
$h_{21} = 24 \qquad h_{22} = 2.5 \times 10^{-4}$ mho

For the same signal source and load as in Problem 23, calculate the voltage gain for this circuit configuration.

25. An air-core transformer with adjustable coupling has a primary self-inductance of 200 mh and a secondary self-inductance of 240 mh. If a 500 μa current at