12.48. In the network shown in Fig. 12-54(*a*) the $50\angle 45^\circ$ source contains a current I. The 10 ohm resistor is changed to 5 ohms. Use the compensation theorem to find V_c and ΔI shown in Fig. 12-54(*b*).
Ans. $21.45\angle -166^\circ$, $2.74\angle -36^\circ$

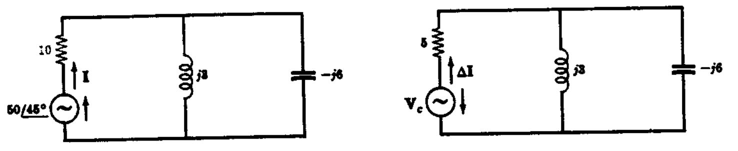

Fig. 12-54

12.49. In the circuit shown in Fig. 12-55 find the value of R_L which results in maximum power transfer. Calculate the value of the maximum power. *Ans.* 11.17 ohms, 309 w

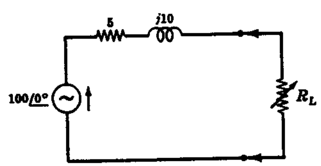

Fig. 12-55

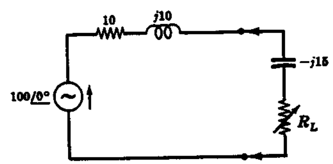

Fig. 12-56

12.50. In the network shown in Fig. 12-56 the load consists of a fixed capacitive reactance of 15 ohms and a variable resistance R_L. Determine (*a*) the value of R_L for which the power transferred is a maximum, (*b*) the value of the maximum power. *Ans.* (*a*) $R_L = 11.17$, (*b*) 236 w

12.51. In the network shown in Fig. 12-57 two voltage sources act on the load impedance connected to the terminals *AB*. If this load is variable in both reactance and resistance, what load Z_L will receive maximum power? What is the value of the maximum power? *Ans.* $(4.23 + j1.15)$, 5.68 w

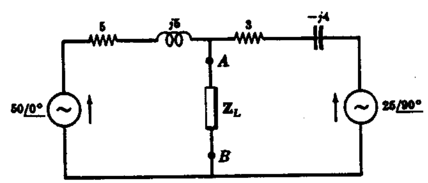

Fig. 12-57

Chapter 13

Mutual Inductance

INTRODUCTION

The networks studied in the previous chapters consisted of loops or meshes and nodes. Since two loops have a common branch and two nodes are joined by either passive or active elements, meshes and nodes are said to be conductively coupled. Methods were developed for the solution of these networks.

In this chapter we analyze another type of coupling, namely, the magnetic coupling. When the interaction between two loops takes place through a magnetic field instead of through common elements, the loops are said to be inductively or magnetically coupled.

SELF-INDUCTANCE

When a current is changing in a circuit, the magnetic flux linking the same circuit changes and an emf is induced in the circuit. Assuming constant permeability, the induced emf is proportional to the rate of change of the current, i.e.,

$$v_L = L\frac{di}{dt} \qquad (1)$$

where the constant of proportionality L is called the self-inductance of the circuit. In the mks system, the unit of self-inductance is the weber/ampere or henry.

In a coil of N turns, the induced emf is given by

$$v_L = N\frac{d\phi}{dt} \qquad (2)$$

where $N\,d\phi$ is the flux linkage of the circuit. Combining equations (1) and (2) we have

$$L\frac{di}{dt} = N\frac{d\phi}{dt}$$

from which

$$L = N\frac{d\phi}{di}$$

MUTUAL INDUCTANCE

Consider a time-varying current i_1 in coil 1 of Fig. 13-1. The changing current i_1 establishes a magnetic flux ϕ_1. Part of this flux links only coil 1 and is called leakage flux ϕ_{11}. The remaining flux ϕ_{12} links also coil 2 as shown. The induced voltage in coil 2 is given by Faraday's law,

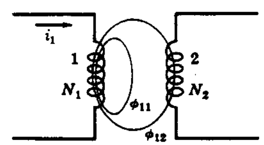

Fig. 13-1

$$v_2 = N_2\frac{d\phi_{12}}{dt} \qquad (3)$$

Since ϕ_{12} is related to the current i_1, v_2 is proportional to the rate of change of i_1, or

$$v_2 \;=\; M\frac{di_1}{dt} \qquad (4)$$

where the constant of proportionality M is called the mutual inductance between the two coils. In the mks system, the unit of mutual inductance is the same as the unit of self-inductance (the henry).

Combining equations (3) and (4), we have

$$v_2 \;=\; N_2\frac{d\phi_{12}}{dt} \;=\; M\frac{di_1}{dt}$$

and

$$M \;=\; N_2\frac{d\phi_{12}}{di_1} \qquad (5)$$

With a set of coils wound on the same iron core, the flux and current are not linearly related and the mutual inductance is given by equation (5). If the coils are linked with air as the medium, the flux and current are linearly related and the mutual inductance is

$$M \;=\; \frac{N_2\phi_{12}}{i_1} \qquad (6)$$

Mutual coupling is bilateral and analogous results are obtained if a time-varying current i_2 is introduced in coil 2 of Fig. 13-1. Then the linking fluxes are ϕ_2, ϕ_{21} and ϕ_{22}, the induced voltage in coil 1 is $v_1 = M(di_2/dt)$ and equations (5) and (6) become respectively

$$(7)\quad M = \frac{N_1\,d\phi_{21}}{di_2} \qquad \text{and} \qquad (8)\quad M = \frac{N_1\phi_{21}}{i_2}$$

COUPLING COEFFICIENT k

In Fig. 13-1 the linkage flux depends on spacing and orientation of the axes of the coils and on the permeability of the medium. The fraction of total flux which links the coils is called the coefficient of coupling k. Then

$$k \;=\; \frac{\phi_{12}}{\phi_1} \;=\; \frac{\phi_{21}}{\phi_2}$$

Since $\phi_{12} \leq \phi_1$ and $\phi_{21} \leq \phi_2$, the maximum value of k is unity.

An expression for M in terms of self-inductances L_1 and L_2 is obtained as follows. Multiply equation (6) by (8) and obtain

$$M^2 \;=\; \left(\frac{N_2\phi_{12}}{i_1}\right)\left(\frac{N_1\phi_{21}}{i_2}\right) \;=\; \left(\frac{N_2k\phi_1}{i_1}\right)\left(\frac{N_1k\phi_2}{i_2}\right) \;=\; k^2\left(\frac{N_1\phi_1}{i_1}\right)\left(\frac{N_2\phi_2}{i_2}\right) \qquad (9)$$

Substituting $L_1 = N_1\phi_1/i_1$ and $L_2 = N_2\phi_2/i_2$ in (9),

$$M^2 \;=\; k^2L_1L_2 \qquad \text{and} \qquad M \;=\; k\sqrt{L_1L_2}$$

ANALYSIS OF COUPLED CIRCUITS

In order to show the winding sense and its effects on the voltages of mutual inductance, the coils are shown on a core as in Fig. 13-2 below.

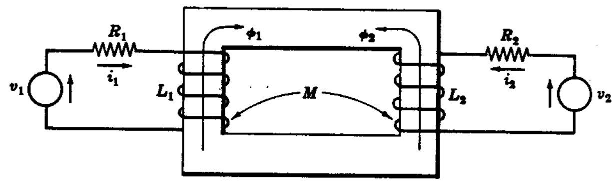

Fig. 13-2

Since each circuit contains a voltage source, select mesh currents i_1 and i_2 in the same direction as the sources and write the two mesh equations according to Kirchhoff's voltage law.

$$\begin{aligned} R_1i_1 + L_1\frac{di_1}{dt} \pm M\frac{di_2}{dt} &= v_1 \\ R_2i_2 + L_2\frac{di_2}{dt} \pm M\frac{di_1}{dt} &= v_2 \end{aligned} \qquad (10)$$

The voltages of mutual inductance may be of either polarity depending on the winding sense. To determine the correct signs in (10) apply the right hand rule to each coil, allowing the fingers to wrap around in the direction of the assumed current. Then the right thumb points in the direction of the flux. Thus the positive directions of ϕ_1 and ϕ_2 are as shown in the figure. *If fluxes ϕ_1 and ϕ_2 due to the assumed positive current directions aid one another, then the signs on the voltages of mutual inductance are the same as the signs on the voltages of self-inductance.* Referring to Fig. 13-2 we note that ϕ_1 and ϕ_2 oppose each other. Then rewriting (10) with the correct signs, we have

$$\begin{aligned} R_1i_1 + L_1\frac{di_1}{dt} - M\frac{di_2}{dt} &= v_1 \\ R_2i_2 + L_2\frac{di_2}{dt} - M\frac{di_1}{dt} &= v_2 \end{aligned} \qquad (11)$$

Assuming sinusoidal voltage sources, the set (11) in the sinusoidal steady state becomes

$$\begin{aligned} (R_1 + j\omega L_1)\mathbf{I}_1 - j\omega M\mathbf{I}_2 &= \mathbf{V}_1 \\ -j\omega M\mathbf{I}_1 + (R_2 + j\omega L_2)\mathbf{I}_2 &= \mathbf{V}_2 \end{aligned} \qquad (12)$$

Recalling the general set of two simultaneous mesh current equations (Chapter 9), we have

$$\begin{aligned} \mathbf{Z}_{11}\mathbf{I}_1 \pm \mathbf{Z}_{12}\mathbf{I}_2 &= \mathbf{V}_1 \\ \pm\mathbf{Z}_{21}\mathbf{I}_1 + \mathbf{Z}_{22}\mathbf{I}_2 &= \mathbf{V}_2 \end{aligned} \qquad (13)$$

We found that $\mathbf{Z}_{12} = \mathbf{Z}_{21}$ were the impedances common to the two mesh currents $\mathbf{I}_1$ and $\mathbf{I}_2$. The meshes were conductively coupled since the currents passed through a common branch. Now in the circuit of Fig. 13-2 we have a similar set of equations where $j\omega M$ corresponds to $\mathbf{Z}_{12}$ and $\mathbf{Z}_{21}$ of equations (13). The meshes are not conductively coupled since the two currents do not have any common impedances. However, the equations indicate that coupling does exist. In such cases the coupling is called *mutual* or *magnetic coupling.*

NATURAL CURRENT

In the preceding section a circuit with two mutually coupled loops, each containing a voltage source, were examined after assuming the directions of the currents. It is necessary at times to analyze the natural current in a loop containing no driving voltages. The direction of this current is determined by application of Lenz's law.

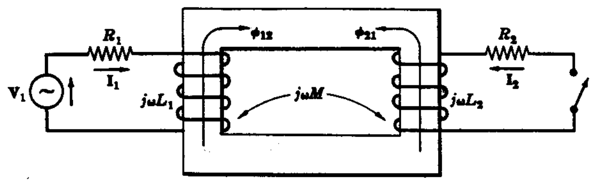

Fig. 13-3

Consider the circuit shown in Fig. 13-3 where only mesh 1 contains a driving voltage. Select current $\mathbf{I}_1$ in agreement with the source $\mathbf{V}_1$ and apply the right hand rule to determine the direction of the flux ϕ_{12}. Now Lenz's law states that the polarity of the induced voltage is such that if the circuit is completed, a current will pass through the coil in a direction which creates a flux opposing the main flux set up by current $\mathbf{I}_1$. Therefore when the switch is closed in the circuit of Fig. 13-3, the direction of flux ϕ_{21} according to Lenz's law is as shown. Now apply the right hand rule with the thumb pointing in the direction of ϕ_{21}; the fingers will wrap around coil 2 in the direction of the *natural current*. Then the mesh current equations are

$$\begin{aligned} (R_1 + j\omega L_1)\mathbf{I}_1 \; - \; j\omega M\mathbf{I}_2 \; &= \; \mathbf{V}_1 \\ -j\omega M\mathbf{I}_1 \; + \; (R_2 + j\omega L_2)\mathbf{I}_2 \; &= \; 0 \end{aligned} \qquad (14)$$

Since mesh 2 has no driving voltage, it follows that the natural current $\mathbf{I}_2$ resulted from the voltage of mutual inductance, $(R_2 + j\omega L_2)\mathbf{I}_2 = (j\omega M\mathbf{I}_1)$. In Fig. 13-4 this voltage is shown as a source. The direction of the source must be as indicated by the arrow for the positive direction of $\mathbf{I}_2$. Hence, *the instantaneous polarity of the voltage of mutual inductance at coil two is positive at the terminal where the natural current leaves the winding.*

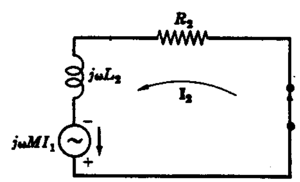

Fig. 13-4

DOT RULE FOR COUPLED COILS

While the relative polarity for voltages of mutual inductance can be determined from sketches of the core which show the winding sense, the method is not practical. To simplify the diagrammatic representation of coupled circuits, the coils are marked with dots as shown in Fig. 13-5(*c*). On each coil, a dot is placed at the terminals which are instantaneously of the same polarity on the basis of the mutual inductance alone. Then to apply the dot notation we must know at which terminal of the coils the dots are assigned. Moreover, we must determine the sign associated with the voltage of mutual inductance when we write the mesh current equations.

To assign the dots on a pair of coupled coils, select a current direction in one coil of the pair and place a dot at the terminal where this current enters the winding. The dotted terminal is instantaneously positive with respect to the other terminal of the coil. Apply the right hand rule to find the corresponding flux as shown in Fig. 13-5(*a*). Now in the second coil the flux must oppose the original flux, according to Lenz's law. See Fig. 13-5(*b*).

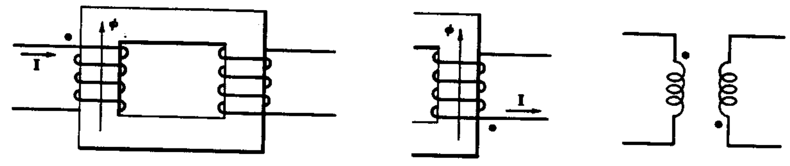

Fig. 13-5

Use the right hand rule to find the direction of the natural current, and since the voltage of mutual inductance is positive at the terminal where this natural current leaves the winding, place a dot at this terminal as shown in Fig. 13-5(*b*). With the instantaneous polarity of the coils given by the dots, the core is no longer needed in the diagram and the coupled coils may be illustrated as in Fig. 13-5(*c*).

To determine the sign of the voltage of mutual inductance in the mesh current equations, we use the dot rule which states: (1) *When both assumed currents enter or leave a pair of coupled coils at the dotted terminals, the signs of the M terms will be the same as the signs of the L terms;* (2) *If one current enters at a dotted terminal and one leaves by a dotted terminal, the signs of the M terms are opposite to the signs of the L terms.*

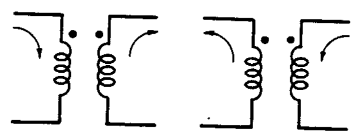

Fig. 13-6

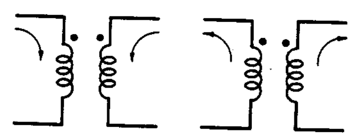

Fig. 13-7

Fig. 13-6 above shows when the signs of the M and L terms are opposite. Fig. 13-7 above shows two cases in which the signs of M and L are the same.

As a further illustration of the relative polarities in connection with mutual coupled circuits, consider the circuit of Fig. 13-8 where the dots are marked and the currents $\mathbf{I}_1$ and $\mathbf{I}_2$ selected as shown. Since one current enters at a dotted terminal and the other leaves by a dotted terminal, the sign on the M terms is opposite to the sign on the L terms. For this circuit the mesh current equations in matrix form are

$$\begin{bmatrix} \mathbf{Z}_{11} & -j\omega M \\ -j\omega M & \mathbf{Z}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} \;=\; \begin{bmatrix} \mathbf{V}_1 \\ 0 \end{bmatrix} \qquad (15)$$

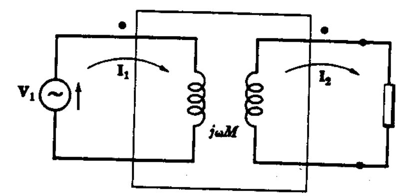

Fig. 13-8

Now a simple conductive coupled two mesh network is given in Fig. 13-9 and the positive terminals are indicated. The mesh current equations in matrix form are

$$\begin{bmatrix} Z_{11} & -Z \\ -Z & Z_{22} \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ 0 \end{bmatrix} \qquad (16)$$

The impedance Z common to both mesh currents shows a negative sign since the currents I_1 and I_2 pass in opposite directions through the branch containing Z.

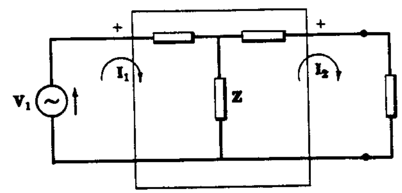

Fig. 13-9

When the boxes in Fig. 13-8 and Fig. 13-9 are covered, the two circuits look identical except for the dot notation in one circuit and the sign notation in the other. Comparing *(15)* and *(16)*, the negative sign of $j\omega M$ corresponds to the negative sign of Z.

CONDUCTIVELY COUPLED EQUIVALENT CIRCUITS

It is possible in analysis to replace a mutually coupled circuit with a conductively coupled equivalent circuit. Consider the circuit of Fig. 13-10(*a*). Select the directions of currents I_1 and I_2 as shown. Then the mesh current equations in matrix form are

$$\begin{bmatrix} R_1 + j\omega L_1 & -j\omega M \\ -j\omega M & R_2 + j\omega L_2 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \qquad (17)$$

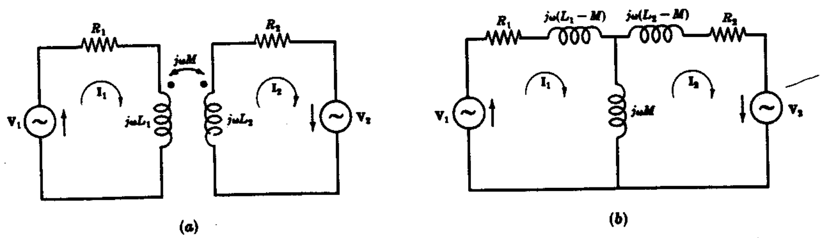

Fig. 13-10

Let the current directions in Fig. 13-10(*b*) be the same as in Fig. 13-10(*a*). The currents I_1 and I_2 pass in opposite directions through the common branch; then the required impedance here is $j\omega M$. In *(17)*, $Z_{11} = R_1 + j\omega L_1$. Since mesh current I_1 passes through the common branch with impedance $j\omega M$, we must insert $(-j\omega M)$ in the loop and write

$$Z_{11} = R_1 + j\omega L_1 - j\omega M + j\omega M = R_1 + j\omega L_1$$

Similarly in loop 2,

$$Z_{22} = R_2 + j\omega L_2 - j\omega M + j\omega M = R_2 + j\omega L_2$$

If we write the mesh current equations for the circuit in Fig. 13-10(*b*) we obtain the set *(17)*. Thus the conductively coupled circuit of Fig. 13-10(*b*) is equivalent to the mutually coupled circuit of Fig. 13-10(*a*).

The above method of analysis does not always lead to a physically realizable equivalent circuit. This is true when $M > L_1$ or $M > L_2$.

To replace the series connection of the mutually coupled coils shown in Fig. 13-11(*a*), proceed in the following manner. First apply the methods described above and obtain the dotted equivalent shown in Fig. 13-11(*b*). Then replace the dotted equivalent by the conductive equivalent shown in Fig. 13-11(*c*).

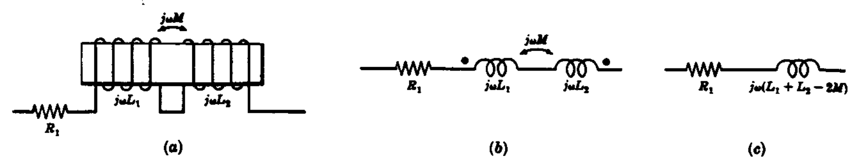

Fig. 13-11

Analysis of the circuit shown in Fig. 13-11(*a*) requires consideration of magnetic fluxes to determine the signs of the voltages of mutual inductance. With the circuit of Fig. 13-11(*b*), no fluxes need be considered but the dot rule is required. Then with the circuit of Fig. 13-11(*c*) the necessary equations can be written in the usual manner with no special attention to either flux, dots or mutual inductance. All three circuits have the same complex impedance $Z = R_1 + j\omega(L_1 + L_2 - 2M)$.

Solved Problems

13.1. Coil 1 of a pair of coupled coils has a continuous current of 5 amp, and the corresponding fluxes ϕ_{11} and ϕ_{12} are 20,000 and 40,000 maxwells respectively. If the turns are $N_1 = 500$ and $N_2 = 1500$, find L_1, L_2, M and k. (1 weber = 10^8 maxwells.)

The total flux is $\phi_1 = \phi_{11} + \phi_{12} = 60{,}000$ maxwells $= 6 \times 10^{-4}$ weber. Then the self-inductance of coil 1 is $L_1 = N_1\phi_1/I_1 = 500(6 \times 10^{-4})/5 = .06$ h.

The coupling coefficient $k = \phi_{12}/\phi_1 = 40{,}000/60{,}000 = .667$.

The mutual inductance $M = N_2\phi_{12}/I_1 = 1500(4 \times 10^{-4})/5 = .12$ h.

Since $M = k\sqrt{L_1L_2}$, $.12 = .667\sqrt{.06L_2}$ and $L_2 = .539$ h.

13.2. Two coupled coils of $L_1 = .8$ h and $L_2 = .2$ h have a coupling coefficient $k = .9$. Find the mutual inductance M and the turns ratio N_1/N_2.

The mutual inductance is $M = k\sqrt{L_1L_2} = .9\sqrt{.8(.2)} = .36$ h.

Using $M = N_2\phi_{12}/i_1$, substitute $k\phi_1$ for ϕ_{12} and multiply by N_1/N_1 to obtain

$$M = k\frac{N_2}{N_1}\left(\frac{N_1\phi_1}{i_1}\right) = k\frac{N_2}{N_1}L_1 \quad \text{and} \quad N_1/N_2 = kL_1/M = .9(.8)/.36 = 2$$

13.3. Two coupled coils with respective self-inductances $L_1 = .05$ h and $L_2 = .20$ h have a coupling coefficient $k = .5$. Coil 2 has 1000 turns. If the current in coil 1 is $i_1 = 5 \sin 400t$, determine the voltage at coil 2 and the maximum flux set up by coil 1.

The mutual inductance is $M = k\sqrt{L_1 L_2} = .5\sqrt{.05(.20)} = .05$ h. Then the voltage at coil 2 is given by $v_2 = M(di_1/dt) = .05\frac{d}{dt}(5 \sin 400t) = 100 \cos 400t$. Since the voltage at coil 2 is also given by $v_2 = N_2(d\phi_{12}/dt)$, we obtain

$$100 \cos 400t = 1000(d\phi_{12}/dt)$$

and
$$\phi_{12} = 10^{-3}\int 100 \cos 400t \, dt = .25 \times 10^{-3} \sin 400t$$

The maximum of the flux ϕ_{12} is $.25 \times 10^{-3}$ weber. Then the maximum of ϕ_1 is

$$\phi_{1\,max} = \frac{\phi_{12\,max}}{.5} = \frac{.25 \times 10^{-3}}{.5} = .5 \times 10^{-3} \text{ weber}$$

13.4. Apply Kirchhoff's voltage law to the coupled circuit shown in Fig. 13-12 and write the equation in instantaneous form.

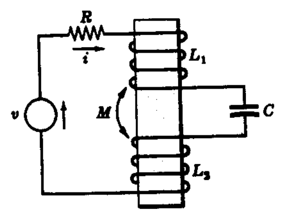

Fig. 13-12

Examination of the winding sense of the coils shows that the signs on the M terms are opposite to the signs on the L terms. Also note that the voltage of mutual inductance appears at each coil due to the current i in the other coil of the pair.

$$Ri + L_1\frac{di}{dt} - M\frac{di}{dt} + \frac{1}{c}\int i\,dt + L_2\frac{di}{dt} - M\frac{di}{dt} = v$$

or
$$Ri + (L_1 + L_2 - 2M)\frac{di}{dt} + \frac{1}{c}\int i\,dt = v$$

13.5. Write the mesh current equations in instantaneous form for the coupled circuit of Fig. 13-13 below.

Select mesh currents i_1 and i_2 as shown in the diagram and apply the right hand rule to each winding. Since the fluxes aid, the sign of the M terms are the same as the sign of the L terms. Then

$$R_1 i_1 + L_1\frac{di_1}{dt} + M\frac{di_2}{dt} = v$$

$$R_2 i_2 + L_2\frac{di_2}{dt} + M\frac{di_1}{dt} = v$$

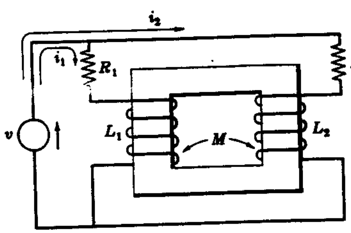

Fig. 13-13

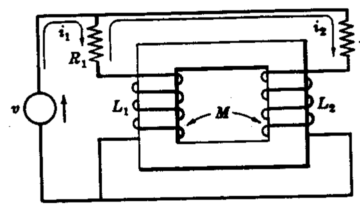

Fig. 13-14

13.6. Repeat Problem 13.5 with mesh current i_2 as shown in Fig. 13-14 above.

In applying Kirchhoff's voltage law to the loop of current i_2, the voltages of mutual inductance are negative. Thus

$$R_1(i_1 - i_2) + L_1\frac{d}{dt}(i_1 - i_2) + M\frac{di_2}{dt} = v$$

$$R_1(i_2 - i_1) + R_2 i_2 + L_2\frac{di_2}{dt} - M\frac{d}{dt}(i_2 - i_1) + L_1\frac{d}{dt}(i_2 - i_1) - M\frac{di_2}{dt} = 0$$

13.7. Two coils connected in series have an equivalent inductance L_A if the connection is aiding and an equivalent inductance L_B if the connection is opposing. Find the mutual inductance M in terms of L_A and L_B.

When the connection is aiding, the equivalent inductance is given by

$$L_A = L_1 + L_2 + 2M \qquad (1)$$

When the connection is opposing, we have

$$L_B = L_1 + L_2 - 2M \qquad (2)$$

Subtracting (2) from (1),

$$L_A - L_B = 4M \quad \text{and} \quad M = \tfrac{1}{4}(L_A - L_B)$$

This solution points out an experimental method for the determination of M. Connect the coils both ways and obtain the equivalent inductances on an AC bridge. The resulting inductance is one-fourth of the difference between the two equivalent inductances.

13.8. Obtain the dotted equivalent circuit for the coupled circuit shown in Fig. 13-15. Find the voltage across the $-j10$ reactance using the equivalent circuit.

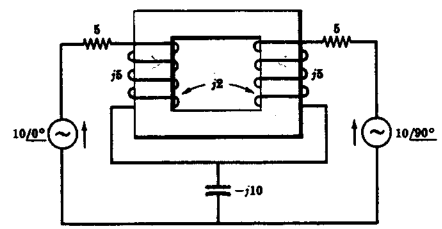

Fig. 13-15

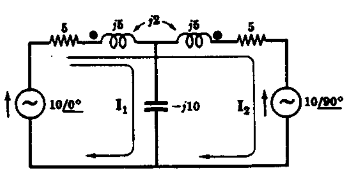

Fig. 13-16

To place the dots on the circuit, consider only the coils and their winding sense. Drive a current into the top of the left coil and place a dot at this terminal. The corresponding flux direction is upward on the left side of the core. By Lenz's law the flux at the right coil must also be upward. Then the right hand rule gives the direction of the natural current. This current leaves the winding by the upper terminal, which should then be marked with a dot as shown in Fig. 13-16.

For i_1 and i_2 selected as shown, the mesh current equations in matrix form are

$$\begin{bmatrix} 5 - j5 & 5 + j3 \\ 5 + j3 & 10 + j6 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 - j10 \end{bmatrix}$$

from which

$$I_1 = \frac{\begin{vmatrix} 10 & 5+j3 \\ 10-j10 & 10+j6 \end{vmatrix}}{\Delta_z} = 1.015\underline{/113.95°}$$

Then the voltage across the $-j10$ reactance is

$$V = I_1(-j10) = 10.15\underline{/23.95°}$$

13.9. Obtain the dotted equivalent for the coupled coils shown in Fig. 13-17, and write the corresponding equation.

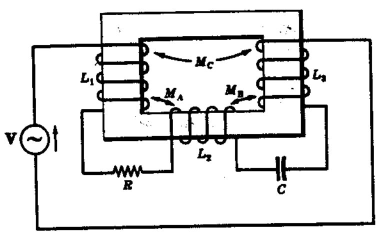

Fig. 13-17

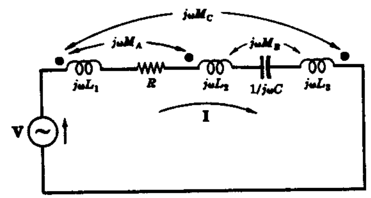

Fig. 13-18

Assign the dots using the methods of Problem 13.8 to obtain the circuit shown in Fig. 13-18. Applying Kirchhoff's voltage law to the single loop,

$$\left[R + \frac{1}{j\omega C} + j\omega(L_1 + L_2 + L_3 + 2M_A - 2M_B - 2M_C)\right] I = V$$

13.10. In the coupled network of Fig. 13-19 find the voltage across the 5 ohm resistor for the dots as given in the diagram. Then reverse the polarity in one coil and repeat.

Compute the mutual inductance from

$$jX_m = jk\sqrt{X_{L1}X_{L2}} = j\,.8\sqrt{5(10)} = j5.66$$

Solving for mesh current I_2,

$$I_2 = \frac{\begin{vmatrix} 3+j1 & 50 \\ -3-j1.66 & 0 \end{vmatrix}}{\begin{vmatrix} 3+j1 & -3-j1.66 \\ -3-j1.66 & 8+j6 \end{vmatrix}} = \frac{171\underline{/29°}}{19.9\underline{/53.8°}} = 8.60\underline{/-24.8°}$$

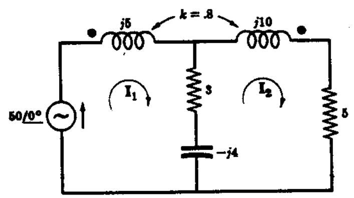

Fig. 13-19

Then the voltage across the 5 ohm resistor is $V_5 = I_2(5) = 43\underline{/-24.8°}$.

With a change in polarity in one coil the impedance matrix changes, resulting in a new value of the mesh current I_2.

$$I_2 = \frac{\begin{vmatrix} 3+j1 & 50 \\ -3+j9.66 & 0 \end{vmatrix}}{\begin{vmatrix} 3+j1 & -3+j9.66 \\ -3+j9.66 & 8+j6 \end{vmatrix}} = \frac{505\underline{/-72.7°}}{132\underline{/39.4°}} = 3.83\underline{/-112.1°}$$

The voltage across the 5 ohm resistor is $V_5 = I_2(5) = 19.15\underline{/-112.1°}$.

13.11. Obtain the equivalent inductance of the parallel connection of L_1 and L_2 shown in Fig. 13-20(a).

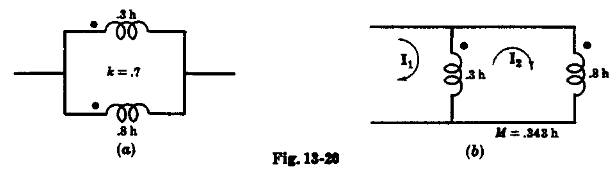

Fig. 13-20

The mutual inductance $M = k\sqrt{L_1L_2} = .7\sqrt{.3(.8)} = .343$ h. Set up the circuit as in Fig. 13-20(b) and insert the mesh currents. Now

$$[Z] = \begin{bmatrix} j\omega .3 & j\omega .043 \\ j\omega .043 & j\omega .414 \end{bmatrix}$$

$$Z_{\text{input 1}} = \frac{\Delta_z}{\Delta_{11}} = \frac{j\omega .3(j\omega .414) - (j\omega .043)^2}{j\omega .414} = j\omega .296$$

Thus the equivalent inductance of the coupled coils is .296 h.

13.12. The Heaviside bridge circuit in Fig. 13-21 is used to determine the mutual inductance between a pair of coils. Find M in terms of the other bridge constants when detector current I_D equals zero.

Select two mesh currents I_1 and I_2 as shown in the diagram. If $I_D = 0$ the voltage drops across R_1 and R_2 must be equal:

$$I_1R_1 = I_2R_2 \qquad (1)$$

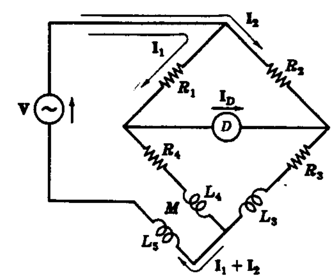

Fig. 13-21

Similarly the drops across $(R_4 + j\omega L_4)$ and $(R_3 + j\omega L_3)$ must be equal. However, at L_4 there appears a voltage of mutual induction, and the current in the other coil of the pair, L_5, is the sum $I_1 + I_2$.

$$I_1(R_4 + j\omega L_4) + j\omega M(I_1 + I_2) = I_2(R_3 + j\omega L_3) \qquad (2)$$

Substituting $I_2 = (R_1/R_2)I_1$ into (2),

$$I_1(R_4 + j\omega L_4 + j\omega M) + (R_1/R_2)I_1(j\omega M) = (R_1/R_2)I_1(R_3 + j\omega L_3) \qquad (3)$$

Equating real and imaginary parts of (3),

$$R_4R_2 = R_1R_3 \quad \text{and} \quad j\omega\left(L_4 + M + \frac{R_1}{R_2}M\right) = j\omega\frac{R_1}{R_2}L_3 \quad \text{from which} \quad M = \frac{R_1L_3 - R_2L_4}{R_1 + R_2}$$

13.13. Replace the coupled network shown in Fig. 13-22 with a Thevenin equivalent at the terminals AB.

The Thevenin equivalent voltage $\mathbf{V}'$ is the open circuit voltage at the terminals AB. Select mesh currents $\mathbf{I}_1$ and $\mathbf{I}_2$ as shown and solve for $\mathbf{I}_2$.

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 5+j5 & 10 \\ -2+j3 & 0 \end{vmatrix}}{\begin{vmatrix} 5+j5 & -2+j3 \\ -2+j3 & 6+j5 \end{vmatrix}} = \frac{20-j30}{10+j67} = .533\underline{/-137.8^\circ}$$

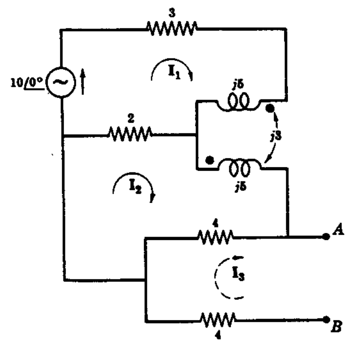

Fig. 13-22

Now $\quad \mathbf{V}' = \mathbf{V}_{AB} = \mathbf{I}_2(4) = 2.13\underline{/-137.8^\circ}$

To determine $\mathbf{Z}'$ of the Thevenin equivalent, set up the third mesh current $\mathbf{I}_3$ and compute $\mathbf{Z}_{\text{input}\,3}$ which is the impedance seen at the terminals AB with all internal sources set to zero.

$$\mathbf{Z}' = \mathbf{Z}_{\text{input}\,3} = \frac{\Delta_z}{\Delta_{33}} = \frac{\begin{vmatrix} 5+j5 & -2+j3 & 0 \\ -2+j3 & 6+j5 & -4 \\ 0 & -4 & 8 \end{vmatrix}}{\begin{vmatrix} 5+j5 & -2+j3 \\ -2+j3 & 6+j5 \end{vmatrix}}$$

$$= \frac{j456}{10+j67} = 6.74\underline{/8.5^\circ}$$

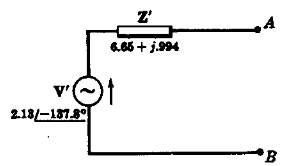

Fig. 13-23

The Thevenin equivalent circuit is shown in Fig. 13-23.

13.14. In the two loop coupled circuit shown in Fig. 13-24, show that the dots are not necessary so long as the second loop is passive.

Select mesh currents as shown in the diagram and solve for $\mathbf{I}_2$.

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 2+j5 & 50 \\ \pm j4 & 0 \end{vmatrix}}{\begin{vmatrix} 2+j5 & \pm j4 \\ \pm j4 & 5+j10 \end{vmatrix}} = \frac{-50(\pm j4)}{-24+j45} = 3.92\underline{/61.9^\circ} \pm 90^\circ$$

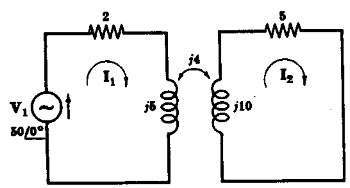

Fig. 13-24

The value of Δ_z is unaffected by the sign on M, and the current $\mathbf{I}_2$ will have a phase angle of either 151.9° or −28.1°. Since there is no voltage source in the loop it is unnecessary that the polarity of the voltage of mutual inductance be known. Voltage drops across the loop impedances would be equal in magnitude and differ in phase by 180°. Power in an impedance would be unaffected. It is also true that $\mathbf{I}_1$ is identical for either sign on the mutual.

13.15. In the circuit shown in Fig. 13-25, find R_L which results in maximum power transfer after selecting the best connection for the coils and finding k.

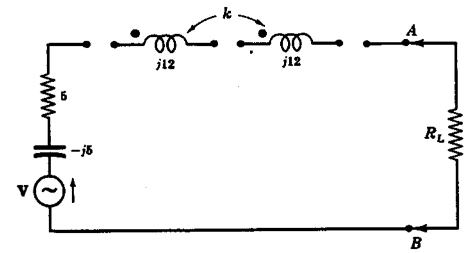

Fig. 13-25

Fig. 13-26

The circuit impedance to the left of AB must be a minimum. Express the impedance of this part of the circuit.

$$\mathbf{Z} = 5 - j5 + j12 + j12 \pm j2X_M = 5 + j19 \pm j2k\sqrt{12(12)}$$

For minimum impedance the reactance should be zero; hence, the proper sign for the mutual is negative.

$$19 - 2k\sqrt{12(12)} = 0 \quad \text{and} \quad k = 19/24 = .792$$

The connection shown in Fig. 13-26 results in a negative sign on the voltages of mutual inductance as required. Then the circuit impedance to the left of AB is 5 ohms pure resistance, and the maximum power results when $R_L = R_g = 5$ ohms.

13.16. The circuit of Fig. 13-25 has a load resistance $R_L = 10$ ohms and a source $\mathbf{V} = 50\underline{/0^\circ}$. With both connections of the coils possible and k variable from 0 to 1, find the range of power that can be delivered to the load resistor.

With the coupling shown in Fig. 13-26 the sign on the mutual is negative and the total circuit impedance including the load is $\mathbf{Z}_T = 5 - j5 + j12 + j12 - j24k + 10$. Set $k = 1$; then

$$\mathbf{Z}_T = 15 - j5 = 15.8\underline{/-18.45^\circ}, \qquad \mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_T} = \frac{50\underline{/0^\circ}}{15.8\underline{/-18.45^\circ}} = 3.16\underline{/18.45^\circ}$$

The power in the 10 ohm resistor is $P = I^2R = (3.16)^2(10) = 100$ watts.

Now set $k = 0$; then

$$\mathbf{Z}_T = 15 + j19 = 24.2\underline{/51.7^\circ}, \qquad \mathbf{I} = 50\underline{/0^\circ}/(24.2\underline{/51.7^\circ}) = 2.06\underline{/-51.7^\circ}$$

The power in the 10 ohm resistor is $P = I^2R = (2.06)^2(10) = 42.4$ watts.

Change the connection of the coils to result in a positive sign of the mutual inductance. Then the impedance becomes $\mathbf{Z}_T = 15 + j19 + jk24$.

Set $k = 1$; then

$$\mathbf{Z}_T = 15 + j43 = 45.6\underline{/70.8^\circ}, \qquad \mathbf{I} = 50\underline{/0^\circ}/(45.6\underline{/70.8^\circ}) = 1.095\underline{/-70.8^\circ}$$

The corresponding power $P = I^2R = (1.095)^2(10) = 12$ watts.

Thus the 10 ohm resistor can expect a power within the range 12 to 100 watts.

13.17. Obtain a conductively coupled equivalent circuit for the mutually coupled circuit shown in Fig. 13-27.

Select mesh currents I_1 and I_2 as shown and write the matrix equation.

$$\begin{bmatrix} 3+j1 & -3-j2 \\ -3-j2 & 8+j6 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 50\angle 0^\circ \\ 0 \end{bmatrix}$$

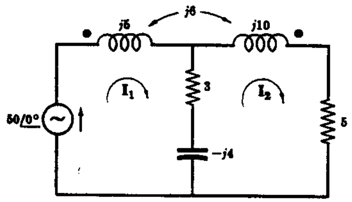

Fig. 13-27

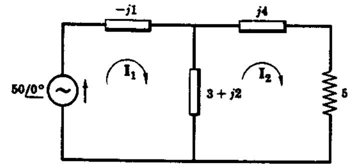

Fig. 13-28

Select the mesh currents in the conductively coupled circuit with the same directions as in the mutually coupled circuit. From the impedance matrix, $Z_{12} = -3 - j2$. Since the currents pass through the common branch in opposite directions, the required branch impedance is $3 + j2$. Now the self-impedance of loop 1 is $Z_{11} = 3 + j1$. Then a $-j1$ impedance is required in the loop. Similarly, since $Z_{22} = 8 + j6$, the loop requires a $5 + j4$ impedance in addition to the elements of the common branch as shown in Fig. 13-28.

13.18. Obtain the conductively coupled equivalent circuit for the mutually coupled network shown in Fig. 13-29.

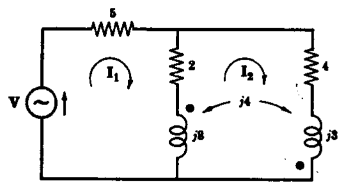

Fig. 13-29

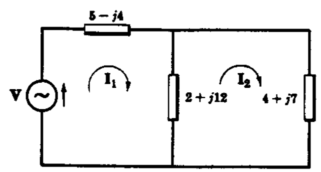

Fig. 13-30

Select the mesh currents I_1 and I_2 and write the mesh current equations in matrix form.

$$\begin{bmatrix} 7+j8 & -2-j12 \\ -2-j12 & 6+j19 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix}$$

The mesh currents in the conductively coupled circuit pass through the common branch in opposite directions. Since Z_{12} in the impedance matrix is $-2 - j12$, the impedance of this branch must be $2 + j12$. Also, from the impedance matrix, $Z_{11} = 7 + j8$ and $Z_{22} = 6 + j19$. Then the remaining impedances in loops 1 and 2 of the equivalent circuit are respectively

$$Z_1 = (7+j8) - (2+j12) = 5 - j4 \quad \text{and} \quad Z_2 = (6+j19) - (2+j12) = 4 + j7$$

The conductively coupled equivalent circuit is shown in Fig. 13-30.

Supplementary Problems

13.19. Two coils have a coupling coefficient $k = .85$ and coil 1 has 250 turns. With a current $i_1 = 2$ amp in coil 1, the total flux ϕ_1 is 3.0×10^{-4} weber. When i_1 is reduced linearly to zero in two milliseconds the voltage induced in coil 2 is 63.75 volts. Find L_1, L_2, M and N_2.
Ans. 37.5 mh, 150 mh, 63.8 mh, 500

13.20. Two coupled coils with turns $N_1 = 100$ and $N_2 = 800$ have a coupling coefficient of .85. With coil 1 open and a current of 5 amperes in coil 2, the flux ϕ_2 is 3.5×10^{-4} weber. Find L_1, L_2 and M.
Ans. .875, 56, 5.95 mh

13.21. If two identical coils have an equivalent inductance of .080 h in series aiding and .035 h in series opposing, what are the values of L_1, L_2, M and k?
Ans. $L_1 = 28.8$ mh, $L_2 = 28.8$ mh, $M = 11.25$ mh, .392

13.22. Two coupled coils with $L_1 = .02$ h, $L_2 = .01$ h and $k = .5$ are connected in four different ways: series aiding, series opposing, and parallel with both arrangements of the winding sense. What are the four equivalent inductances? *Ans.* 15.9, 44.1, 9.47, 3.39 mh

13.23. Two identical coils with $L = .02$ h have a coupling coefficient $k = .8$. Find M and the two equivalent inductances with the coils connected in series aiding and series opposing.
Ans. 16, 72, 8 mh

13.24. Two coils with inductances in the ratio of four to one have a coupling coefficient $k = .6$. When these coils are connected in series aiding the equivalent inductance is 44.4 mh. Find L_1, L_2 and M.
Ans. 6, 24, 7.2 mh

13.25. Two coils with inductances $L_1 = 6.8$ mh and $L_2 = 4.5$ mh are connected in series aiding and series opposing. The equivalent inductances of these connections are 19.6 and 3 mh respectively. Find M and k.
Ans. 4.15 mh, .75

13.26. Select mesh currents for the coupled circuit of Fig. 13-31 and write the equations in instantaneous form. Obtain the dotted equivalent circuit, write the equations and compare the results.

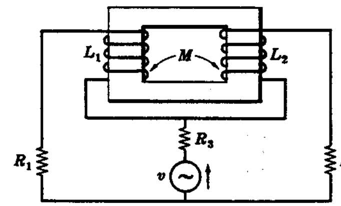

Fig. 13-31

13.27. Sketch the dotted equivalent circuit for the coupled coils shown in Fig. 13-32 and find the equivalent inductive reactance. *Ans.* $j12$

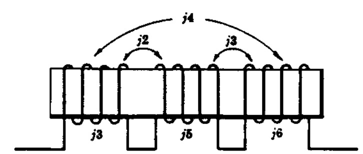

Fig. 13-32

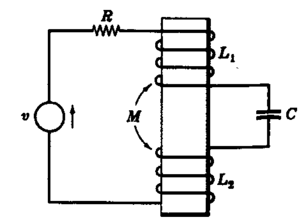

Fig. 13-33

13.28. Obtain the dotted equivalent circuit for the coupled coils of Fig. 13-33 and write the equation in instantaneous form.

13.29. Sketch the dotted equivalent circuit for the coupled coils shown in Fig. 13-34 and find the current I. *Ans.* $4.47\underline{/26.7^\circ}$

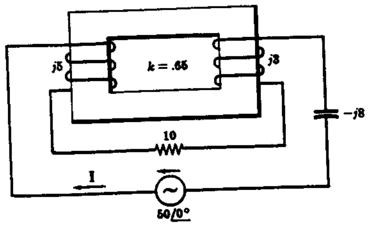

Fig. 13-34

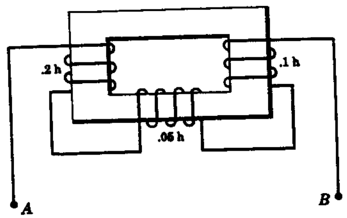

Fig. 13-35

13.30. Obtain the dotted equivalent circuit for the three coupled coils shown in Fig. 13-35 and find the equivalent inductance at the terminals AB. All coupling coefficients are .5. *Ans.* .239 h

13.31. Obtain the dotted equivalent circuit for the coupled circuit of Fig. 13-36 and find the equivalent impedance at the terminals AB. *Ans.* $2.54 + j2.26$

13.32. Referring to the coupled circuit of Fig. 13-36, reverse the winding of one coil and find the equivalent impedance. *Ans.* $2.53 + j\,.238$

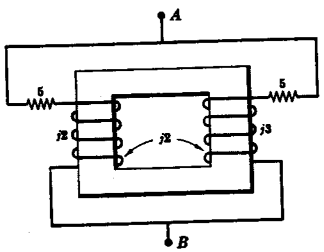

Fig. 13-36

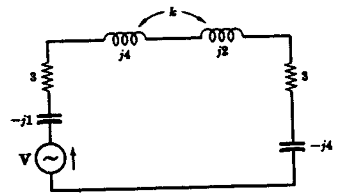

Fig. 13-37

13.33. For the series circuit shown in Fig. 13-37 find the value of k and place the dots on the coupled coils such that the circuit is in series resonance. *Ans.* $k = .177$

13.34. For the series circuit of Fig. 13-38 find k and place the dots such that the circuit is in series resonance. *Ans.* $k = .112$

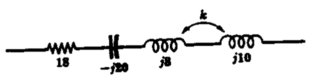

Fig. 13-38

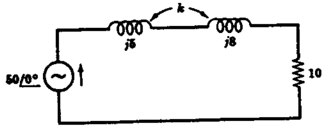

Fig. 13-39

13.35. For the circuit shown in Fig. 13-39 find k and place the dots so that the power output of the $50\underline{/0^\circ}$ volt source is 168 watts. *Ans.* $k = .475$

13.36. Referring to Problem 13.35, find the power output of the source when the dots are reversed. Use the value of k found in Problem 13.35. *Ans.* 54.2 w

13.37. For the coupled circuit shown in Fig. 13-40, find the voltage ratio V_2/V_1 which results in zero current I_1. Repeat for a zero current I_2. *Ans.* $1.414\underline{/-45^\circ}$, $.212\underline{/32^\circ}$

13.38. Referring to Problem 13.37 what voltage appears across the $j8$ reactance when V_1 is $100\underline{/0^\circ}$ and $I_1 = 0$? *Ans.* $100\underline{/0^\circ}$ (+ at dot)

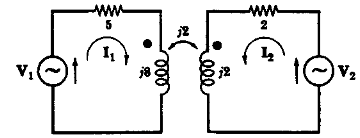

Fig. 13-40

13.39. In the coupled circuit of Fig. 13-41 find the mutual inductive reactance $j\omega M$ if the power in the 5 ohm resistor is 45.2 watts. *Ans.* $j4$

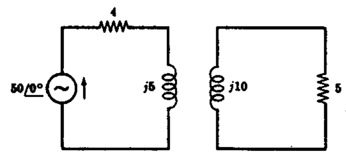

Fig. 13-41

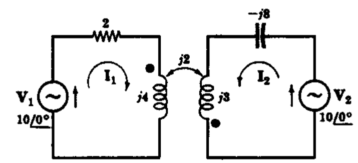

Fig. 13-42

13.40. For the coupled circuit of Fig. 13-42 find the components of current I_2 caused by each source V_1 and V_2. *Ans.* $.77\underline{/112.6^\circ}$, $1.72\underline{/86^\circ}$

13.41. Determine the value of k in the coupled circuit of Fig. 13-43 if the power in the 10 ohm resistor is 32 watts. *Ans.* .791

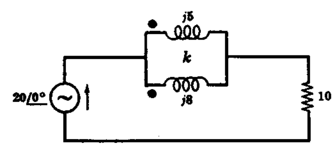

Fig. 13-43

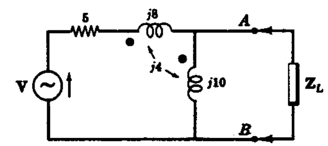

Fig. 13-44

13.42. For the circuit shown in Fig. 13-44 find the load impedance Z_L which results in maximum power transfer across the terminals AB. *Ans.* $1.4 - j2.74$

13.43. For the coupled circuit shown in Fig. 13-45 find the input impedance at the terminals of the source. *Ans.* $3 + j36.3$

13.44. Referring to the circuit of Fig. 13-45, find the voltage across the $j5$ reactance if the source $V = 50\underline{/45^\circ}$. *Ans.* $25.2\underline{/49.74^\circ}$

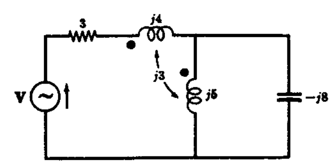

Fig. 13-45

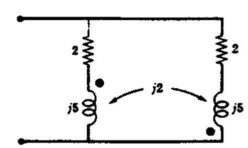

Fig. 13-46

13.45. Find the equivalent impedance of the coupled circuit shown in Fig. 13-46. *Ans.* [illegible] $+ j1.5$

13.46. Obtain the Thevenin equivalent circuit at terminals AB of the coupled circuit given in Fig. 13-47.
Ans. $Z' = 2 + j6.5$, $V' = 5 + j5$

13.47. Referring to the coupled network of Fig. 13-47, obtain the Norton equivalent circuit at terminals AB.
Ans. $Z' = 2 + j6.5$, $I' = 1.04\underline{/-27.9^\circ}$

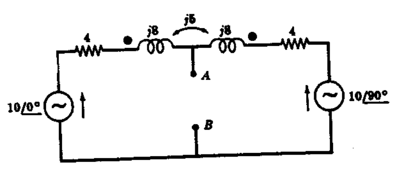

Fig. 13-47

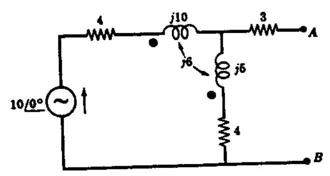

Fig. 13-48

13.48. Obtain the Thevenin equivalent circuit at terminals AB of the coupled circuit shown in Fig. 13-48.
Ans. $Z' = 8.63\underline{/48.75^\circ}$, $V' = 4.84\underline{/-34.7^\circ}$

13.49. Find the Norton equivalent circuit for the coupled network of Fig. 13-48.
Ans. $Z' = 8.63\underline{/48.75^\circ}$, $I' = .560\underline{/-83.4^\circ}$

13.50. For the coupled circuit shown in Fig. 13-49 find the input impedance at the terminals of the voltage source V. *Ans.* $7.06 + j3.22$

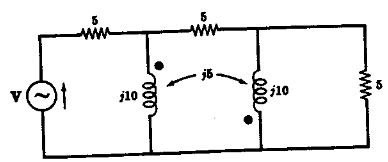

Fig. 13-49

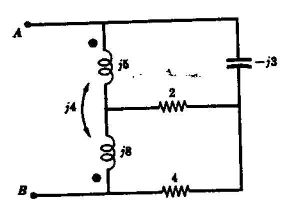

Fig. 13-50

13.51. Find the equivalent impedance at terminals AB of the coupled network of Fig. 13-50.
Ans. $6.22 + j4.65$

Chapter 14

Polyphase Systems

INTRODUCTION

A polyphase system consists of two or more equal voltages with fixed phase differences which supply power to loads connected to the lines. In the two-phase system two equal voltages differ in phase by 90°, while in the three-phase system the voltages have a phase difference of 120°. Systems of six or more phases are sometimes used in polyphase rectifiers to obtain a rectified voltage with low ripple, but three-phase is the system commonly used for generation and transmission of electric power.

TWO-PHASE SYSTEM

Rotation of the pair of perpendicular coils in Fig. 14-1(a) in the constant magnetic field results in induced voltages with a fixed 90° phase difference. With equal number of turns of the coils, the phasor and instantaneous voltages have equal magnitudes as shown in their respective diagrams in Figures 14-1(b) and (c).

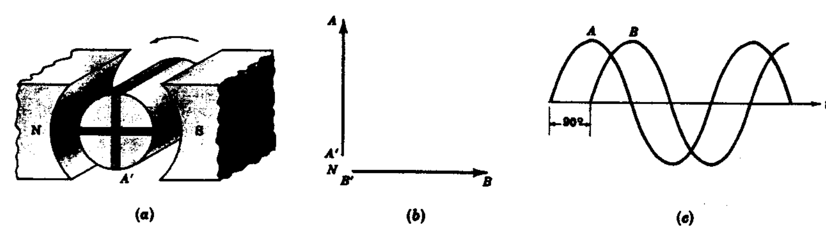

Fig. 14-1. Two-phase System

The voltage phasor diagram in Fig. 14-1(b) has as a reference $\mathbf{V}_{BN} = V_{coil}\underline{/0^\circ}$ and the voltage $\mathbf{V}_{AN} = V_{coil}\underline{/90^\circ}$. If the coil ends A' and B' are joined as line N, the two-phase system is contained on the three lines A, B and N. The potential between lines A and B exceeds the line to neutral voltages by the factor $\sqrt{2}$ and is obtained from the sum, $\mathbf{V}_{AB} = \mathbf{V}_{AN} + \mathbf{V}_{NB} = V_{coil}\underline{/90^\circ} + V_{coil}\underline{/180^\circ} = \sqrt{2}\,V_{coil}\underline{/135^\circ}$.

THREE-PHASE SYSTEM

The induced voltages in the three equally spaced coils in Fig. 14-2(a) below have a phase difference of 120°. The voltage in coil A reaches a maximum first, followed by B and then C for sequence ABC. This sequence is evident from the phasor diagram with its positive rotation counterclockwise where the phasors would pass a fixed point in the order A-B-C-A-B-$C\cdots$, and also from the instantaneous voltage plot of Fig. 14-2(c) below where the maxima occur in the same order.