Chapter 13

Mutual Inductance

INTRODUCTION

The networks studied in the previous chapters consisted of loops or meshes and nodes. Since two loops have a common branch and two nodes are joined by either passive or active elements, meshes and nodes are said to be conductively coupled. Methods were developed for the solution of these networks.

In this chapter we analyze another type of coupling, namely, the magnetic coupling. When the interaction between two loops takes place through a magnetic field instead of through common elements, the loops are said to be inductively or magnetically coupled.

SELF-INDUCTANCE

When a current is changing in a circuit, the magnetic flux linking the same circuit changes and an emf is induced in the circuit. Assuming constant permeability, the induced emf is proportional to the rate of change of the current, i.e.,

$$v_L = L \frac{di}{dt}$$

where the constant of proportionality $L$ is called the self-inductance of the circuit. In the mks system, the unit of self-inductance is the weber/ampere or henry.

In a coil of $N$ turns, the induced emf is given by

$$v_L = N \frac{d\phi}{dt}$$

where $N \phi$ is the flux linkage of the circuit. Combining equations (1) and (2) we have

$$L \frac{di}{dt} = N \frac{d\phi}{dt}$$

from which

$$L = N \frac{d\phi}{dt}$$

MUTUAL INDUCTANCE

Consider a time-varying current $i$ in coil 1 of Fig. 13-1. The changing current $i$ establishes a magnetic flux $\phi_1$. Part of this flux links only coil 1 and is called leakage flux $\phi_{leak}$. The remaining flux $\phi_{int}$ links also coil 2 as shown. The induced voltage in coil 2 is given by Faraday's law.

$$v_2 = N_2 \frac{d\phi_{int}}{dt}$$

Fig. 13-1
Since \( \phi_{i2} \) is related to the current \( i_2 \), \( v_2 \) is proportional to the rate of change of \( i_2 \), or

\[
v_2 = M \frac{di_2}{dt}
\]

(4)

where the constant of proportionality \( M \) is called the mutual inductance between the two coils. In the mks system, the unit of mutual inductance is the same as the unit of self-inductance (the henry).

Combining equations (3) and (4), we have

\[
v_1 = N_1 \frac{d\phi_{i1}}{dt} = M \frac{di_2}{dt}
\]

and

\[
M = N_2 \frac{d\phi_{i2}}{di_1}
\]

(6)

With a set of coils wound on the same iron core, the flux and current are not linearly related and the mutual inductance is given by equation (5). If the coils are linked with air as the medium, the flux and current are linearly related and the mutual inductance is

\[
M = \frac{N_2 \phi_{i2}}{i_1}
\]

(6)

Mutual coupling is bilateral and analogous results are obtained if a time-varying current \( i_1 \) is introduced in coil 1 of Fig. 13-1. Then the linking fluxes are \( \phi_1, \phi_2, \) and \( \phi_{i1}, \) the induced voltage in coil 1 is \( v_1 = M(di_1/dt) \) and equations (3) and (6) become respectively

\[
(7) \quad M = \frac{N_1 \phi_{i1}}{i_1}
\]

and

\[
(8) \quad M = \frac{N_2 \phi_{i2}}{i_1}
\]

COUPLING COEFFICIENT \( k \)

In Fig. 13-1 the linkage flux depends on spacing and orientation of the axes of the coils and on the permeability of the medium. The fraction of total flux which links the coils is called the coefficient of coupling \( k \). Then

\[
k = \frac{\phi_{i2}}{\phi_{i1}} = \frac{\phi_{i2}}{\phi_{i1}}
\]

Since \( \phi_{i2} = \phi_2 \) and \( \phi_{i1} = \phi_1 \), the maximum value of \( k \) is unity.

An expression for \( M \) in terms of self-inductances \( L_1 \) and \( L_2 \) is obtained as follows. Multiply equation (6) by (6) and obtain

\[
M^2 = \left( \frac{N_2 \phi_{i1}}{i_1} \right) \left( \frac{N_1 \phi_{i2}}{i_2} \right) = \left( \frac{N_2 \phi_{i1}}{i_1} \right) \left( \frac{N_1 \phi_{i2}}{i_2} \right) = k^2 \left( \frac{N_1 \phi_1}{i_1} \right) \left( \frac{N_2 \phi_2}{i_2} \right)
\]

(9)

Substituting \( L_1 = N_1 \phi_1/i_1 \) and \( L_2 = N_2 \phi_2/i_2 \) in (9),

\[
M^2 = k^2 \frac{L_1 L_2}{L_1 L_2}
\]

and

\[
M = k\sqrt{L_1 L_2}
\]

ANALYSIS OF COUPLED CIRCUITS

In order to show the winding sense and its effects on the voltages of mutual inductance, the coils are shown on a core as in Fig. 13-2 below.

**Fig. 13-2**

Since each circuit contains a voltage source, select mesh currents \( i_1 \) and \( i_2 \) in the same direction as the sources and write the two mesh equations according to Kirchhoff's voltage law.

\[
R_{i1}i_1 + L_{i1} \frac{di_1}{dt} = M \frac{di_2}{dt} = v_1
\]

(10)

\[
R_{i2}i_2 + L_{i2} \frac{di_2}{dt} = M \frac{di_1}{dt} = v_2
\]

The voltages of mutual inductance may be of either polarity depending on the winding sense. To determine the correct signs in (10) apply the right hand rule to each coil, allowing the fingers to wrap around in the direction of the assumed current. Then the right thumb points in the direction of the flux. Thus the positive directions of \( \phi_1 \) and \( \phi_2 \) are as shown in the figure. If fluxes \( \phi_1 \) and \( \phi_2 \) due to the assumed positive current directions aid one another, then the signs on the voltages of mutual inductance are the same as the signs on the voltages of self-inductance. Referring to Fig. 13-2 we note that \( \phi_1 \) and \( \phi_2 \) oppose each other. Then rewriting (10) with the correct signs, we have

\[
R_{i1}i_1 + L_{i1} \frac{di_1}{dt} = M \frac{di_2}{dt} = v_1
\]

(11)

\[
R_{i2}i_2 + L_{i2} \frac{di_2}{dt} = -M \frac{di_1}{dt} = v_2
\]

(12)

Assuming sinusoidal voltage sources, the set (11) in the sinusoidal steady state becomes

\[
(R_1 + j\omega L_{i1})i_1 - j\omega M_1i_2 = V_1
\]

\[
-j\omega M_1i_2 + (R_1 + j\omega L_{i1})i_1 = V_2
\]

Recalling the general set of two simultaneous mesh current equations (Chapter 9), we have

\[
Z_{i1}i_1 + Z_{i1}i_2 = V_1
\]

(13)

\[
Z_{i2}i_2 + Z_{i2}i_1 = V_2
\]

(14)

We found that \( Z_{i1} = Z_{i1} \) were the impedances common to the two mesh currents \( i_1 \) and \( i_2 \). The meshes were conductively coupled since the currents passed through a common branch. Now in the circuit of Fig. 13-2 we have a similar set of equations where \( j\omega M \) corresponds to \( Z_{i1} \) and \( Z_{i2} \) of equations (13). The meshes are not conductively coupled since the two currents do not have any common impedances. However, the equations indicate that coupling does exist. In such cases the coupling is called mutual or magnetic coupling.
NATURAL CURRENT

In the preceding section a circuit with two mutually coupled loops, each containing a voltage source, were examined after assuming the directions of the currents. It is necessary at times to analyze the natural current in a loop containing no driving voltages. The direction of this current is determined by application of Lenz’s law.

Consider the circuit shown in Fig. 13-3 where only mesh 1 contains a driving voltage. Select current $I_1$ in agreement with the source $V_1$, and apply the right hand rule to determine the direction of the flux $\phi_{\alpha}$. Now Lenz’s law states that the polarity of the induced voltage is such that if the circuit is completed, a current will pass through the coil in a direction which creates a flux opposing the main flux set up by current $I_1$. Therefore when the switch is closed in the circuit of Fig. 13-3, the direction of flux $\phi_{\alpha}$, according to Lenz’s law is as shown. Now apply the right hand rule with the thumb pointing in the direction of $\phi_{\alpha}$; the fingers will wrap around coil 2 in the direction of the natural current. Then the mesh current equations are

\[ (R_1 + j\omega L_1)I_1 - j\omega M I_2 = V_1 \]
\[ -j\omega M I_1 + (R_2 + j\omega L_2)I_2 = 0 \]  
(14)

Since mesh 2 has no driving voltage, it follows that the natural current $I_2$ resulted from the voltage of mutual inductance, $(R_2 + j\omega L_2)I_2 = j\omega M I_1$. In Fig. 13-4 this voltage is shown as a source. The direction of the source must be indicated by the arrow for the positive direction of $I_2$. Hence, the instantaneous polarity of the voltage of mutual inductance at coil two is positive at the terminal where the natural current leaves the winding.

DOT RULE FOR COUPLED COILS

While the relative polarity for voltages of mutual inductance can be determined from sketches of the core which show the winding sense, the method is not practical. To simplify the diagrammatic representation of coupled circuits, the coils are marked with dots as shown in Fig. 13-6(c). On each coil, a dot is placed at the terminals which are instantaneously of the same polarity on the basis of the mutual inductance alone. Then to apply the dot notation we must know at which terminal of the coils the dots are assigned. Moreover, we must determine the sign associated with the voltage of mutual inductance when we write the mesh current equations.

To assign the dots on a pair of coupled coils, select a current direction in one coil of the pair and place a dot at the terminal where this current enters the winding. The dotted terminal is instantaneously positive with respect to the other terminal of the coil. Apply the right-hand rule to find the corresponding flux as shown in Fig. 13-6(a). Now in the second coil the flux must oppose the original flux, according to Lenz’s law. See Fig. 13-6(b).

Use the right hand rule to find the direction of the natural current, and since the voltage of mutual inductance is positive at the terminal where this natural current leaves the winding, place a dot at this terminal as shown in Fig. 13-6(b). With the instantaneous polarity of the coils given by the dots, the core is no longer needed in the diagram and the coupled coils may be illustrated as in Fig. 13-6(c).

To determine the sign of the voltage of mutual inductance in the mesh current equations, we use the dot rule which states: (1) When both assumed currents enter or leave a pair of coupled coils at the dotted terminals, the signs of the M terms will be the same as the signs of the L terms; (2) If one current enters at a dotted terminal and one leaves by a dotted terminal, the signs of the M terms are opposite to the signs of the L terms.

As a further illustration of the relative polarities in connection with mutual coupled circuits, consider the circuit of Fig. 13-7 where the dots are marked and the currents $I_1$ and $I_2$ are selected as shown. Since one current enters at a dotted terminal and the other leaves by a dotted terminal, the sign on the $M$ terms is opposite to the sign on the $L$ terms. For this circuit the mesh current equations in matrix form are

\[ \begin{bmatrix} Z_{11} & -j\omega M \\ -j\omega M & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ 0 \end{bmatrix} \]  
(15)
Now a simple conductive coupled two mesh network is given in Fig. 13-9 and the positive terminals are indicated. The mesh current equations in matrix form are

\[
\begin{bmatrix}
  Z_{11} & -Z & 1 & I_1 \\
  -Z & Z_{22} & I_2 \\
  1 & I_2 \\
\end{bmatrix}
\begin{bmatrix}
  V_1 \\
  V_2 \\
  0
\end{bmatrix}
\]

(16)

The impedance \( Z \) common to both mesh currents shows a negative sign since the currents \( I_1 \) and \( I_2 \) pass in opposite directions through the branch containing \( Z \).

When the boxes in Fig. 13-8 and Fig. 13-9 are covered, the two circuits look identical except for the dot notation in one circuit and the sign notation in the other. Comparing (15) and (16), the negative sign of \( j\omega M \) corresponds to the negative sign of \( Z \).

CONDUCTIVELY COUPLED EQUIVALENT CIRCUITS

It is possible in analysis to replace a mutually coupled circuit with a conductively coupled equivalent circuit. Consider the circuit of Fig. 13-10(a). Select the directions of currents \( I_1 \) and \( I_2 \) as shown. Then the mesh current equations in matrix form are

\[
\begin{bmatrix}
  R_1 + j\omega L_1 & -j\omega M & I_1 \\
  -j\omega M & R_2 + j\omega L_2 & I_2 \\
\end{bmatrix}
\begin{bmatrix}
  V_1 \\
  V_2 \\
\end{bmatrix}
\]

(17)

Let the current directions in Fig. 13-10(b) be the same as in Fig. 13-10(a). The currents \( I_1 \) and \( I_2 \) pass in opposite directions through the common branch; then the required impedance here is \( j\omega M \). In (17), \( Z_{11} = R_1 + j\omega L_1 \). Since mesh current \( I_1 \) passes through the common branch with impedance \( j\omega M \), we must insert \( -j\omega M \) in the loop and write

\[
Z_{11} = R_1 + j\omega L_1 - j\omega M + j\omega M = R_1 + j\omega L_1
\]

Similarly in loop 2,

\[
Z_{22} = R_2 + j\omega L_2 - j\omega M + j\omega M = R_2 + j\omega L_2
\]

If we write the mesh current equations for the circuit in Fig. 13-10(b) we obtain the set (17). Thus the conductively coupled circuit of Fig. 13-10(b) is equivalent to the mutually coupled circuit of Fig. 13-10(a).

Solved Problems

13.1. Coll 1 of a pair of coupled coils has a continuous current of 5 amp, and the corresponding fluxes \( \phi_1 \) and \( \phi_2 \) are 20,000 and 40,000 maxwells respectively. If the turns are \( N_1 = 500 \) and \( N_2 = 1500 \), find \( L_1, L_2, M \) and \( k \). (1 weber = 10,000 maxwells.)

The total flux is \( \phi_1 = \phi_2 = 60,000 \) maxwells = \( 6 \times 10^{-4} \) weber. Then the self-inductance of coil 1 is \( L_1 = N_1^2/\phi_1 = 500(6 \times 10^{-4})/5 = .06 \) h.

The coupling coefficient \( k = \phi_2/\phi_1 = .0600/60,000 = .001 \).

The mutual inductance \( M = N_2\phi_2/\phi_1 = 1500(6 \times 10^{-4})/5 = .12 \) h.

Since \( M = k\sqrt{L_1 L_2} .12 = .0600/\sqrt{500 \times 500} \) and \( L_2 = .539 \) h.

13.2. Two coupled coils of \( L_1 = .8 \) h and \( L_2 = .2 \) h have a coupling coefficient \( k = .9 \). Find the mutual inductance \( M \) and the turns ratio \( N_2/N_1 \).

The mutual inductance is \( M = k\sqrt{L_1 L_2} = .9\sqrt{.8 \times .2} = .36 \) h.

Using \( M = N_2\phi_2/\phi_1 \), substitute \( \phi_2 \) for \( \phi_1 \) and multiply by \( N_1/N_2 \) to obtain

\[
M = \frac{N_2^2/\phi_1}{N_1^2/\phi_1} = k\frac{N_2^2}{N_1^2} \quad \text{and} \quad \frac{N_2}{N_1} = kL_2/M = .9/.36 = 2
\]
13.3. Two coupled coils with respective self-inductances \( L_1 = 0.05 \text{ h} \) and \( L_2 = 0.20 \text{ h} \) have a coupling coefficient \( k = 0.5 \). Coil 2 has 1000 turns. If the current in coil 1 is \( i_1 = 3 \sin 400t \), determine the voltage at coil 2 and the magnetic flux set up by coil 1.

The mutual inductance is \( M = k\sqrt{L_1 L_2} = 0.05 \text{ h} \). Then the voltage at coil 2 is given by \( v_2 = M(\frac{di_1}{dt}) = 0.05 \frac{\sin 400t}{5 \sin 400t} = 0.1 \text{ volts} \). Since the voltage at coil 2 is also given by \( v_2 = N_2(\frac{di_2}{dt}) \), we obtain

\[
100 \cos 400t = 1000(\frac{di_2}{dt})
\]

and

\[
\phi_{23} = 10^{-9} \int 100 \cos 400t \, dt = 25 \times 10^{-9} \text{ weber}
\]

The maximum of the flux \( \phi_{23} \) is \( 25 \times 10^{-9} \text{ weber} \). Then the maximum of \( \phi_1 \) is

\[
\phi_{1\text{max}} = \frac{\phi_{23} \text{max}}{5} = \frac{25 \times 10^{-9}}{5} = 5 \times 10^{-9} \text{ weber}
\]

13.4. Apply Kirchhoff’s voltage law to the coupled circuit shown in Fig. 13-12 and write the equation in instantaneous form.

Examination of the winding sense of the coils shows that the signs on the \( M \) terms are opposite to the signs on the \( L \) terms. Also note that the voltage of mutual inductance appears at each coil due to the current \( i \) in the other coil of the pair.

\[
R_i \left( \frac{di}{dt} \right) + L_1 \frac{di}{dt} = \frac{1}{c} \int i \, dt + L_2 \frac{di}{dt} - M \frac{di}{dt} = v
\]

or

\[
R_i \left( \frac{di}{dt} \right) + (L_1 + L_2 - 2M) \frac{di}{dt} = \frac{1}{c} \int i \, dt = v
\]

13.5. Write the mesh current equations in instantaneous form for the coupled circuit of Fig. 13-13 below.

Select mesh currents \( i_1 \) and \( i_2 \) as shown in the diagram and apply the right-hand rule to each winding. Since the fluxes aid, the sign of the \( M \) terms are the same as the sign of the \( L \) terms. Then

\[
R_i i_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} = v
\]

\[
R_i i_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} = v
\]

13.6. Repeat Problem 13.5 with mesh current \( i_3 \) as shown in Fig. 13-14 above.

In applying Kirchhoff’s voltage law to the loop of current \( i_3 \), the voltages of mutual inductance are negative. Thus

\[
R_i (i_1 - i_3) + L_1 \frac{di_1}{dt} (i_1 - i_3) + M \frac{di_2}{dt} = v
\]

\[
R_i (i_2 - i_3) + R_i i_3 + L_2 \frac{di_2}{dt} - M \frac{di_1}{dt} (i_2 - i_3) + L_1 \frac{di_1}{dt} (i_2 - i_3) - R_i \frac{di_3}{dt} = 0
\]

13.7. Two coils connected in series have an equivalent inductance \( L_a \) if the connection is aiding and an equivalent inductance \( L_a \) if the connection is opposing. Find the mutual inductance \( M \) in terms of \( L_a \) and \( L_b \).

When the connection is aiding, the equivalent inductance is given by

\[
L_a = L_1 + L_2 + 2M
\]

When the connection is opposing, we have

\[
L_a = L_1 + L_2 - 2M
\]

Subtracting (5) from (4),

\[
L_a - L_a = 4M \quad \text{and} \quad M = \frac{1}{4}(L_a - L_a)
\]

This solution points out an experimental method for the determination of \( M \). Connect the coils both ways and obtain the equivalent inductances on an AC bridge. The resulting inductance is one-fourth of the difference between the two equivalent inductances.

13.8. Obtain the dotted equivalent circuit for the coupled circuit shown in Fig. 13-15. Find the voltage across the \(-j10 \text{ reactance using the equivalent circuit.}\)

To place the dots on the circuit, consider only the coils and their winding sense. Drive a current into the top of the left coil and place a dot at this terminal. The corresponding flux direction is upward on the left side of the core. By Lenz’s law the flux at the right coil must also be upward. Then the right-hand rule gives the direction of the natural current. This current leaves the winding by the upper terminal, which should then be marked with a dot as shown in Fig. 13-16.

For \( i_1 \) and \( i_2 \) selected as shown, the mesh current equations in matrix form are

\[
\begin{bmatrix}
5 - j5 & 5 + j5 & i_1 \\
5 + j5 & 10 + j5 & i_2
\end{bmatrix}
= \begin{bmatrix}
10 \\
10 - j10
\end{bmatrix}
\]
13.3. Obtain the dotted equivalent for the coupled coils shown in Fig. 13-17, and write the corresponding equation.

**Fig. 13-17**

Assign the dots using the methods of Problem 13.8 to obtain the circuit shown in Fig. 13-18. Applying Kirchhoff's voltage law to the single loop,

\[ R + j \omega L_c = j \omega M_1 + j \omega M_2 = j \omega M_2 + j \omega M_1 \]

13.10. In the coupled network of Fig. 13-19 find the voltage across the 5 ohm resistor for the dots as given in the diagram. Then reverse the polarity in one coil and repeat.

**Fig. 13-19**

Compute the mutual inductance from

\[ jM_0 = jB \sqrt{L_1 L_2} = 0.8 \sqrt{10(15)} = j5.66 \]

Solving for mesh current \( I_p \),

\[
I_p = \begin{bmatrix} 3 + j1 & 50 \\ -3 - j1.56 & 0 \\ -3 - j1.56 & 0 \end{bmatrix} \]

Then the voltage across the 5 ohm resistor is \( V = I_p(5) = 42.2-j34.8 \).

With a change in polarity in one coil the impedance matrix changes, resulting in a new value of the mesh current \( I_p \).

13.11. Obtain the equivalent inductance of the parallel connection of \( L_1 \) and \( L_2 \) shown in Fig. 13-20(a).

**Fig. 13-20**

The mutual inductance \( M = \frac{L_1 L_2}{L_1 + L_2} = 7 \sqrt{3} \) is \( 3.43 \) h. Set up the circuit as in Fig. 15-20(b) and insert the mesh currents. Now

\[
\begin{bmatrix} 3 & 1 \\
1 & 3 \end{bmatrix} \]

Thus the equivalent inductance of the coupled coils is \( 2.96 \) h.

13.12. The Heaviside bridge circuit in Fig. 13-21 is used to determine the mutual inductance between a pair of coils. Find \( M \) in terms of the other bridge constants when detector current \( I_0 \) equals zero.

**Fig. 13-21**

Select two mesh currents \( I_1 \) and \( I_2 \) as shown in the diagram. If \( I_0 = 0 \) the voltage drops across \( R_1 \) and \( R_2 \) must be equal:

\[ I_1 R_1 = I_2 R_2 \]

Similarly the drops across \( (R_1 + j\omega L_1) \) and \( (R_2 + j\omega L_2) \) must be equal. However, at \( I_0 \) there appears a voltage of mutual induction, and the current in the other coil of the pair, \( I_0 \), is the sum \( I_1 + I_2 \).

\[ (R_1 + j\omega L_1) + j\omega M_1(I_1 + I_2) = I_0(R_1 + j\omega L_1) \]

Substituting \( I_0 = (R_1/R_2)I_1 \) into (3),

\[ (R_1 + j\omega L_1 + j\omega M)(R_1 + j\omega L_1) = (R_1/R_2)I_1(R_1 + j\omega L_1) \]

Equating real and imaginary parts of (3),

\[ R_1 R_2 = R_1 R_2 \]

from which \( M = \frac{R_1 L_2 - R_2 L_1}{R_1 + R_2} \).
13.13. Replace the coupled network shown in Fig. 13-22 with a Thevenin equivalent at the terminals $AB$.

The Thevenin equivalent voltage $V'$ is the open circuit voltage at the terminals $AB$. Select mesh currents $I_1$ and $I_2$ as shown and solve for $I_3$.

\[
I_3 = \begin{bmatrix} 5 + j5 & 10 \\ -2 + j3 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = 20 - j30 \\
\frac{10 + j7}{35} \approx -137.8^\circ
\]

Now $V' = V_{AB} = I_3(4) = 2.13 - j137.8^\circ$

To determine $Z'$ of the Thevenin equivalent, set up the third mesh current $I_3$ and compute $Z_{mesh}$, which is the impedance seen at the terminals $AB$ with all internal sources set to zero.

\[
Z' = \frac{V'}{I_3} = \frac{4.63 + j1.94}{\frac{10 + j7}{35}} \approx 6.74/8.4^\circ
\]

The Thevenin equivalent circuit is shown in Fig. 13-22.

13.14. In the two loop coupled circuit shown in Fig. 13-24, show that the dots are not necessary so long as the second loop is passive.

Select mesh currents as shown in the diagram and solve for $I_2$.

\[
I_3 = \begin{bmatrix} 2 + j5 & 50 \\ \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{50}{\frac{1}{4} + j5} = 3.92/21.9^\circ \approx 90^\circ
\]

The value of $I_2$ is unaffected by the sign on $M$, and the current $I_3$ will have a phase angle of either $15.9^\circ$ or $-28.1^\circ$. Since there is no voltage source in the loop it is unnecessary that the polarity of the voltage of mutual inductance be known. Voltage drops across the loop impedances would be equal in magnitude and differ in phase by $180^\circ$. Power in an impedance would be unaffected. It is also true that $I_3$ is identical for either sign on the mutual.

13.15. In the circuit shown in Fig. 13-25, find $R_e$ which results in maximum power transfer after selecting the best connection for the coils and finding $k$.

\[
Z = 5 - j6 + j12 + j12 = j2X_e = 5 + j19 \approx j2(\sqrt{19/13})
\]

For minimum impedance the reactance should be zero; hence, the proper sign for the mutual is negative.

\[
19 - 2k(19/13) = 0 \quad \text{and} \quad k = 19(13)^{-1} \approx 0.702
\]

The connection shown in Fig. 13-26 results in a negative sign on the voltages of mutual inductances as required. Then the circuit impedance to the left of $AB$ is 5 ohms pure resistance, and the maximum power results when $R_1 = R_2 = 5$ ohms.

13.16. The circuit of Fig. 13-26 has a load resistance $R_e = 10$ ohms and a source $V = 50/0^\circ$. With both connections of the coils possible and $k$ variable from 0 to 1, find the range of power that can be delivered to the load resistor.

With the coupling shown in Fig. 13-26 the sign on the mutual is negative and the total circuit impedance including the load is $Z_T = 5 - j6 + j12 + j12 - 2jAk + 10$. Set $k = 1$, then

\[
Z_T = 15 - j6 = 15.8/18.45^\circ, \quad I = \frac{V}{Z_T} = \frac{50/0^\circ}{15.8/18.45^\circ} = 3.16/18.45^\circ
\]

The power in the 10 ohm resistor is $P = PR = (3.16)(10)$ = 100 watts. Now set $k = 0$, then

\[
Z_T = 15 + j19 = 24.2/81.7^\circ, \quad I = \frac{50/0^\circ}{24.2/81.7^\circ} = 2.06/81.7^\circ
\]

The power in the 10 ohm resistor is $P = PR = (2.06)(10) = 20.6$ watts.

Change the connection of the coils to result in a positive sign of the mutual inductance. Then the impedance becomes $Z_T = 15 + j19 + j24$.

Set $k = 1$, then

\[
Z_T = 15 + j43 = 45.6/70.8^\circ, \quad I = \frac{50/0^\circ}{45.6/70.8^\circ} = 1.095/-70.8^\circ
\]

The corresponding power $P = PR = (1.095)(10)$ = 12 watts.

Thus the 10 ohm resistor can expect a power within the range 12 to 100 watts.
13.17. Obtain a conductively coupled equivalent circuit for the mutually coupled circuit shown in Fig. 13-27.

Select mesh currents \( I_1 \) and \( I_2 \) as shown and write the mesh equation:

\[
\begin{bmatrix}
3 + j1 & -3 - j2 & I_1 \\
-3 - j2 & 8 + j1 & I_2 \\
\end{bmatrix} = \begin{bmatrix}
30/\Omega \\
0 \\
\end{bmatrix}
\]

![Fig. 13-27](image)

![Fig. 13-28](image)

Select the mesh currents in the conductively coupled circuit with the same directions as in the mutually coupled circuit. From the impedance matrix, \( Z_{12} = -3 + j2 \). Since the currents pass through the common branch in opposite directions, the required branch impedance is \( 3 + j2 \). Now the self-impedance of loop 1 is \( Z_{11} = 3 + j1 \). Then a \( -j2 \) impedance is required in the loop. Similarly, since \( Z_{22} = 8 + j6 \), the loop requires a \( 5 + j4 \) impedance in addition to the elements of the common branch as shown in Fig. 13-28.

13.18. Obtain the conductively coupled equivalent circuit for the mutually coupled network shown in Fig. 13-29.

![Fig. 13-29](image)

![Fig. 13-30](image)

Select the mesh currents \( I_1 \) and \( I_2 \) and write the mesh current equations in matrix form:

\[
\begin{bmatrix}
3 + j8 & -5 + j12 & I_1 \\
-2 - j12 & 6 + j19 & I_2 \\
\end{bmatrix} = \begin{bmatrix}
15/\Omega \\
0 \\
\end{bmatrix}
\]

The mesh currents in the conductively coupled circuit pass through the common branch in opposite directions. Since \( Z_{11} \) in the impedance matrix is \( -2 + j12 \), the impedance of this branch must be \( 2 + j12 \). Also, from the impedance matrix, \( Z_{12} = 7 + j6 \) and \( Z_{22} = 6 + j19 \). Then the remaining impedances in loops 1 and 2 of the equivalent circuit are respectively:

\[
Z_1 = (7 + j6) - (2 + j12) = 5 - j4 \quad \text{and} \quad Z_2 = (6 + j19) - (2 + j12) = 4 + j7
\]

The conductively coupled equivalent circuit is shown in Fig. 13-30.

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**Supplementary Problems**

13.19. Two coils have a coupling coefficient \( k = 0.85 \) and coil 1 has 250 turns. With a current \( i_1 = 2 \) amp in coil 1, the total flux \( \phi_1 \) is \( 3.0 \times 10^{-4} \) weber. When \( i_1 \) is reduced linearly to zero in two milliseconds the voltage induced in coil 2 is 63.75 volts. Find \( L_1, L_2, M \) and \( N_2 \).

**Ans.** 27.6 mh, 160 mh, 63.75 volt-mils, 600

13.20. Two coupled coils with turns \( N_1 = 100 \) and \( N_2 = 600 \) have a coupling coefficient of 0.85. With coil 1 open and a current of 5 amperes in coil 2, the flux \( \phi_2 \) is \( 5.5 \times 10^{-4} \) weber. Find \( L_1, L_2, M \) and \( N_2 \).

**Ans.** 375, 56, 8.05 mh

13.21. If two identical coils have an equivalent inductance of 0.080 h in series aiding and 0.05 h in series opposing, what are the values of \( L_2, L_2, M \) and \( k \)?

**Ans.** \( L_1 = 28.8 \) mh, \( L_2 = 28.8 \) mh, \( M = 11.25 \) mh, 0.392

13.22. Two coupled coils with \( L_1 = 0.02 \) h, \( L_2 = 0.01 \) h and \( k = 0.5 \) are connected in four different ways: series aiding, series opposing, parallel with both arrangements of the winding sense. What are the four equivalent inductances?

**Ans.** 15.9, 44.1, 47.8, 5.29 mh

13.23. Two identical coils with \( L_1 = 0.02 \) h have a coupling coefficient \( k = 0.5 \). Find \( M \) and the two equivalent inductances with the coils connected in series aiding and series opposing.

**Ans.** 16, 72, 8 mh

13.24. Two coils with inductances in the ratio of four to one have a coupling coefficient \( k = 0.5 \). When these coils are connected in series aiding the equivalent inductance is 44.4 mh. Find \( L_1, L_2 \) and \( M \).

**Ans.** 6, 24, 7.2 mh

13.25. Two coils with inductances \( L_1 = 6.8 \) mh and \( L_2 = 4.5 \) mh are connected in series aiding and series opposing. The equivalent inductance of these connections are 18.6 and 3 mh respectively. Find \( M \) and \( k \).

**Ans.** 4.15 mh, 0.75

13.26. Select mesh currents for the coupled circuit of Fig. 13-31 and write the equations in instantaneous form. Obtain the dotted equivalent circuit, write the equations and compare the results.

13.27. Sketch the dotted equivalent circuit for the coupled coils shown in Fig. 13-32 and find the equivalent inductive reactance.

**Ans.** \( j12 \)

13.28. Obtain the dotted equivalent circuit for the coupled coils of Fig. 13-33 and write the equation in instantaneous form.
13.23. Sketch the dotted equivalent circuit for the coupled coils shown in Fig. 13-34 and find the current $I_1$.

Ans. $4.47/28.7^\circ$

![Fig. 13-34](image)

![Fig. 13-35](image)

13.24. Obtain the dotted equivalent circuit for the three coupled coils shown in Fig. 13-36 and find the equivalent inductance at the terminals $AB$. All coupling coefficients are $k$. Ans. 2.56

13.25. Obtain the dotted equivalent circuit for the coupled circuit of Fig. 13-36 and find the equivalent impedance at the terminals $AB$.

Ans. $2.54 + j2.56$

13.26. Referring to the coupled circuit of Fig. 13-36, reverse the winding of one coil and find the equivalent impedance.

Ans. $2.53 + j2.58$

![Fig. 13-36](image)

![Fig. 13-37](image)

13.27. For the series circuit shown in Fig. 13-37 find the value of $k$ and place the dots on the coupled coils such that the circuit is in series resonance.

Ans. $k = .177$

13.28. For the series circuit of Fig. 13-38 find $k$ and place the dots such that the circuit is in series resonance.

Ans. $k = .112$

![Fig. 13-38](image)

![Fig. 13-39](image)

13.29. For the circuit shown in Fig. 13-39 find $k$ and place the dots so that the power output of the 50/75$^\circ$ volt source is 160 watts.

Ans. $k = .73$

![Fig. 13-40](image)

![Fig. 13-41](image)

![Fig. 13-42](image)

![Fig. 13-43](image)

![Fig. 13-44](image)

13.30. Referring to Problem 13.30, find the power output of the source when the dots are reversed. Use the value of $k$ found in Problem 13.30.

Ans. 54.2 w

13.31. For the coupled circuit shown in Fig. 13-40, find the voltage ratio $V_2/V_1$ which results in zero current $I_1$. Repeat for a zero current $I_2$.

Ans. $1.414 / -45^\circ$, $212/202^\circ$

13.32. Referring to Problem 13.31 what voltage appears across the /$\beta$ reactance when $V_1$ is 100/90$^\circ$ and $I_1 = 07$

Ans. 100/90$^\circ$ (at dot)

13.33. In the coupled circuit of Fig. 13-41 find the mutual inductive reactance $\omega M$ if the power in the 5 ohm resistor is 45.2 watts.

Ans. $\omega M$

![Fig. 13-41](image)

![Fig. 13-42](image)

13.34. For the coupled circuit of Fig. 13-42 find the components of current $I_2$ caused by each source $V_1$ and $V_2$. Ans. $77/128.6^\circ$, $1.72/86.6^\circ$

13.35. Determine the value of $k$ in the coupled circuit of Fig. 13-43 if the power in the 10 ohm resistor is 32 watts.

Ans. $791$

![Fig. 13-43](image)

![Fig. 13-44](image)

13.36. For the circuit shown in Fig. 13-44 find the load impedance $Z_L$ which results in maximum power transfer across the terminals $AB$.

Ans. $1.4 - j2.74$

13.37. For the coupled circuit shown in Fig. 13-45 find the input impedance at the terminals of the source.

Ans. $2 + 0.3$

13.38. For the circuit of Fig. 13-46, find the voltage across the /$\beta$ reactance if the source $V = 50/65^\circ$.

Ans. $25.2/49.74^\circ$

![Fig. 13-45](image)

![Fig. 13-46](image)

13.39. Find the equivalent impedance of the coupled circuit shown in Fig. 13-46.

Ans. $\omega M$

![Fig. 13-46](image)
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Chapter 14

Polyphase Systems

INTRODUCTION

A polyphase system consists of two or more equal voltages with fixed phase differences which supply power to loads connected to the lines. In the two-phase system two equal voltages differ in phase by 90°, while in the three-phase system the voltages have a phase difference of 120°. Systems of six or more phases are sometimes used in polyphase rectifiers to obtain a rectified voltage with low ripple, but three-phase is the system commonly used for generation and transmission of electric power.

TWO-PHASE SYSTEM

Rotation of the pair of perpendicular coils in Fig. 14-1(a) in the constant magnetic field results in induced voltages with a fixed 90° phase difference. With equal number of turns of the coils, the phasor and instantaneous voltages have equal magnitudes as shown in their respective diagrams in Figures 14-1(b) and (c).

Fig. 14-1. Two-phase System

The voltage phasor diagram in Fig. 14-1(b) has as a reference $V_{AE} = V_{am}/0^\circ$ and the voltage $V_{AB} = V_{am}/90^\circ$. If the coil ends $A'$ and $B'$ are joined as line $N$, the two-phase system is contained on the three lines $A$, $B$ and $N$. The potential between lines $A$ and $B$ exceeds the line to neutral voltages by the factor $\sqrt{2}$ and is obtained from the sum, $V_{AB} = V_{AE} + V_{EB} = V_{am}/90^\circ + V_{am}/180^\circ = \sqrt{2} V_{am}/135^\circ$.

THREE-PHASE SYSTEM

The induced voltages in the three equally spaced coils in Fig. 14-2(a) below have a phase difference of 120°. The voltage in coil $A$ reaches a maximum first, followed by $B$ and then $C$ for sequence $ABC$. This sequence is evident from the phasor diagram with its positive rotation counterclockwise where the phasors would pass a fixed point in the order $A-B-C-A-B-C \cdots$, and also from the instantaneous voltage plot of Fig. 14-2(c) below where the maxima occur in the same order.