Semi-deterministic relay channels with non-causal states

Ido B. Gattegno, Haim H. Permuter, Shlomo Shamai, Ayfer Ö zgür

Ben-Gurion, Technion and Stanford

ICSEE International Conference on the Science of Electrical Engineering
November 16th, 2016
Relay channels are simple models for multi-hop wireless network.
Motivation

Relay channels are simple models for multi-hop wireless network.

States models interferences.
Motivation

- Relay channels are simple models for multi-hop wireless network.
- States models interferences.
- Coding schemes are interesting.
Semi-deterministic Relay channels with non-causal states

\[ Z_i = f(X_i, X_{r,i}, S_i) \] - deterministic link.
Semi-deterministic Relay channels with non-causal states

- \( Z_i = f(X_i, X_{r,i}, S_i) \) - deterministic link.
- Non-causal states.
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- $Z_i = f(X_i, X_{r,i}, S_i)$ - deterministic link.
- Non-causal states.

Theorem (Capacity of SD-RC with non-causal states)

$$C = \max \min \left\{ I(X, X_r; Y | S), I(X; Y | X_r, U, Z, S) + H(Z | X_r, U, S) - I(U; S) \right\}$$

maximum for $P_{U|S}P_{X_r|U}P_{X|X_r,U,S}$ s.t. $I(U; S) < H(Z | X_r, U, S)$. 

Ido B. Gattegno

SD-RC with non-causal CSI

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Semi-deterministic relay channels:

<table>
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<tr>
<th>States</th>
<th>PMF</th>
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<tr>
<td>None</td>
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- Coding scheme with states: cooperative-bin-forward.
One message, both transmitter and relay access the channel.
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Transmitter sends the message.
Semi-deterministic Relay channels

- One message, both transmitter and relay access the channel.
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- Relay observes *deterministic* output.
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- Relay’s cooperate based on past observations.
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C = \max_{P_{X, X_r}} \min \{ I(X, X_r; Y), H(Z|X_r) + I(X; Y|X_r, Z) \}
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\]
Partial-decode forward

- Block-Markov.
Partial-decode forward

- Block-Markov.
- Rate splitting: $m^{(b)} = (m'^{(b)}, m''^{(b)})$. 
Partial-decode forward

- Block-Markov.
- Rate splitting: $m^{(b)} = (m^{(b)}', m^{(b)''})$.
- After block $(b-1)$, the relay decodes $\hat{m}^{(b-1)}$
Partial-decode forward

- Block-Markov.
- Rate splitting: \( m^{(b)} = (m'^{(b)}, m''^{(b)}) \).
- After block \((b - 1)\), the relay decodes \( \hat{m}'^{(b-1)} \)

\[
\begin{array}{cccccc}
  b = 1 & b = 2 & b = 3 & \cdots & b = B \\
  \hat{m}'^{(1)} & \hat{m}'^{(2)} & & & \\
\end{array}
\]

- Relay sends \( x_r^{n(b)}(\hat{m}'^{(b-1)}) \).
Partial-decode forward

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- Rate splitting: $m^{(b)} = (m'^{(b)}, m''^{(b)})$.
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<td>$\hat{m}'^{(1)}$</td>
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- Relay sends $x_r^{n(b)}(\hat{m}'^{(b-1)})$.
- Encoder choose $z^{n(b)}(m'^{(b)} | x_r^{n(b)})$. 

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Partial-decode forward

- Block-Markov.
- Rate splitting: \( m^{(b)} = (m'(b), m''(b)) \).
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\[
\begin{array}{cccccc}
  b &= 1 & b &= 2 & b &= 3 & \ldots & b &= B \\
  \hat{m}'(1) & \\
  \hat{m}'(2) & \\
\end{array}
\]

- Relay sends \( x_n^{(b)}(\hat{m}'^{(b-1)}) \).
- Encoder choose \( z^{n(b)}(m'(b) | x_r^{n(b)}) \).
- Encoder sends \( x^n(m''(b) | m'(b), x_r^{n(b)}(\hat{m}'^{(b-1)})) \).
Partial-decode forward

- Block-Markov.
- Rate splitting: $m^{(b)} = (m'^{(b)}, m''^{(b)})$.
- After block $(b - 1)$, the relay decodes $\hat{m}'^{(b-1)}$

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\begin{array}{cccccc}
  b = 1 & b = 2 & b = 3 & \cdots & b = B \\
  \hat{m}'^{(1)} & \hat{m}'^{(2)} & \cdots & \cdots & \cdots \\
\end{array}
\]

- Relay sends $x^n_r^{(b)}(\hat{m}'^{(b-1)})$.
- Encoder choose $z^n(b) (m'^{(b)} | x^n_r^{(b)})$.
- Encoder sends $x^n (m''^{(b)} | m'^{(b)}, x^n_r^{(b)}(\hat{m}'^{(b-1)}))$.
- Cooperation achieved: $P_{X, X_r}$.
Causal states
Causal states at the Encoder and Decoder only.
Semi-deterministic relay channels with causal states

- Causal states at the Encoder and Decoder only.
- Deterministic output: $Z_i = f(X_i, X_{r,i}, S_i)$.
Semi-deterministic relay channels with causal states

- Causal **states** at the Encoder and Decoder only.
- Deterministic output: \( Z_i = f(X_i, X_{r,i}, S_i) \).
- Decoding at the relay reduces rate.
Causal **states** at the Encoder and Decoder only.

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Decoding at the relay reduces rate.

**How to cooperate?**
Causal **states** at the Encoder and Decoder only.

Deterministic output: \( Z_i = f(X_i, X_{r,i}, S_i) \).

Decoding at the relay reduces rate.

**How to cooperate?**

Cooperative-bin-forward!
Semi-deterministic relay channels with causal states

- Define mapping: $l^{(b)} = \text{bin}(z^{n(b)}) \sim \text{Unif}[1 : 2^{n\tilde{R}}]$
Semi-deterministic relay channels with causal states

- Define mapping: \( l^{(b)} = \text{bin}(z^{n(b)}) \sim \text{Unif}[1 : 2^{n\tilde{R}}] \)
- No decoding at the relay: \( x_r^{n(b)}(l^{(b-1)}) \).

\[
\begin{array}{cccccc}
 b = 1 & b = 2 & b = 3 & \ldots & b = B \\
 l^{(1)} & l^{(2)} & \ldots & \end{array}
\]
Semi-deterministic relay channels with causal states

- Define mapping: \( l^{(b)} = \text{bin}(z^{n(b)}) \sim \text{Unif}[1 : 2^{n\tilde{R}}] \)
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\( l^{(1)} \) \( l^{(2)} \)

- Encoding procedure remains (rate split, block Markov).
Semi-deterministic relay channels with causal states

- Define mapping: $l^{(b)} = \text{bin}(z^{n(b)}) \sim \text{Unif}[1 : 2^{nR}]$
- No decoding at the relay: $x^{n(b)}_{r}(l^{(b-1)})$.

- Encoding procedure remains (rate split, block Markov).
- Cooperation achieved: $P_{X_r}P_{X|X_r,S}$.
Semi-deterministic relay channels with causal states

- Define mapping: $l^{(b)} = \text{bin}(z^{n(b)}) \sim \text{Unif}[1 : 2^{n\tilde{R}}]$
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- Encoding procedure remains (rate split, block Markov).
- Cooperation achieved: $P_{X_r} P_{X|X_r,S}$.

Theorem (SD-RC with causal states, Kolte, Ö zgür, Permuter, 2016 )

\[
C = \max_{P_{X_r} P_{X|X_r,S}} \min \{ I(X, X_r; Y|S), H(Z|X_r, S) + I(X; Y|X_r, Z, S) \}
\]
Non-causal states
States are **non-causal**.
States are **non-causal**.

$X_r$ and $S$ are correlated.
States are **non-causal**.

- $X_r$ and $S$ are correlated.

**Theorem (Capacity of SD-RC with non-causal states)**

\[
C = \max \min \left\{ I(X, X_r; Y | S), I(X; Y | X_r, U, Z, S) + H(Z | X_r, U, S) - I(U; S) \right\}
\]

*maximum for $P_{U|S}P_{X_r|U}P_X|_{X_r, U, S}$, $I(U; S) < H(Z | X_r, U, S)$.**
States are non-causal.

$X_r$ and $S$ are correlated.

**Theorem (Capacity of SD-RC with non-causal states)**

$$C = \max \min \left\{ I(X, X_r; Y|S), I(X; Y|X_r, U, Z, S) + H(Z|X_r, U, S) - I(U; S) \right\}$$

Maximum for $P_{U|S}P_{X_r|U}P_X|X_r, U, S$, $I(U; S) < H(Z|X_r, U, S)$. 
SD-RC with non-causal states

Diagram:

- Encoder
  - $M \rightarrow X_i$
  - $S^n \downarrow$
- Relay
  - $Z_i \leftrightarrow X_{r,i}$
  - $S^n \downarrow$
- Decoder
  - $P_{Y,Z|X,X_r,S}$
  - $Y_i \rightarrow \hat{M}$
Cooperative binning.
1. Cooperative binning.
2. Typicality look-up ([Gel’fand and Pinsker, 1980])
SD-RC with non-causal states

1. Cooperative binning.
2. **Typicality look-up ([Gel’fand and Pinsker, 1980])**
Non-causal states

- [Gel’fand and Pinsker, 1980]
Non-causal states

- [Gel’fand and Pinsker, 1980]

Each message is associated with multiple sequences
Non-causal states

- [Gel’fand and Pinsker, 1980]

Each message is associated with multiple sequences

\[ u^n(m, l) \sim \prod_{i=1}^{n} P_U(u_i(m, l)), \quad m \in [1 : 2^{nR}], \quad l \in [1 : 2^{n\tilde{R}}] \]
Non-causal states

- [Gel’fand and Pinsker, 1980]

\[
M \xrightarrow{\text{Encoder}} X_i \xrightarrow{P_{Y|X,S}} Y_i \xrightarrow{\text{Decoder}} \hat{M}
\]

- Each message is associated with multiple sequences

\[
u^n(m, l) \sim \prod_{i=1}^{n} P_U(u_i(m, l)), \quad m \in [1 : 2^nR], \ l \in [1 : 2^{n\tilde{R}}]
\]

Covering lemma

If \( \tilde{R} > I(U; S) \), then for every \( m \),

\[
\lim_{n \to \infty} \Pr \left\{ (U^n(m, l), S^n) \notin A^e_n(P_U, S) \text{ for all } l \in [1 : 2^{n\tilde{R}}] \right\} = 0
\]
Non-causal cooperative binning

\[ M \rightarrow \text{Encoder} \rightarrow P_{Y,Z|X,X_r,S} \rightarrow \text{Decoder} \rightarrow \hat{M} \]
Non-causal cooperative binning

- Block-Markov.
Non-causal cooperative binning

- Block-Markov.
- Rate splitting:

\[ M \rightarrow (M', M''), \quad R = R' + R'' \]
Non-causal cooperative binning

- Block-Markov.
- Rate splitting:

  \[ M \rightarrow (M', M'') \], \quad R = R' + R''

- Cooperative binning:

  \[ \forall z^n \in \mathcal{Z}^n, \text{bin}(z^n) \sim \mathcal{U}[1 : 2^{nR_U}] \]
Non-causal cooperative binning

- Auxiliary codewords: \( \{u^n(l)\}_{l=1}^{2^{nR_U}} \).
Non-causal cooperative binning

- Auxiliary codewords: \( \{ u^n(l) \}_{l=1}^{2^nRU} \).
- Relay codewords: \( x_r^{n(b)} \sim P(x_r^{n(b)} | u^{n(b)}) (l^{(b-1)}) \).
Non-causal cooperative binning

- **Auxiliary codewords:** \( \{u^n(l)\}_{l=1}^{2^{nR_U}} \).
- **Relay codewords:** \( x_r^{n(b)} \sim P(x_r^{n(b)}|u^{(n(b))}(l^{(b-1)})) \).
- **Encoder looks for correlation with states**

\[
\begin{array}{cccc}
m' & z^n & \text{bin}(z^n) & u^n \\
1 & & & \begin{cases} u^n(1) \in A^n_{\epsilon}(P_{U,S}|s^{n(b+1)}) \\
\quad u^n(2) \notin A^n_{\epsilon}(P_{U,S}|s^{n(b+1)}) \\
\quad u^n(3) \notin A^n_{\epsilon}(P_{U,S}|s^{n(b+1)}) \\
\quad u^n(4) \in A^n_{\epsilon}(P_{U,S}|s^{n(b+1)}) \end{cases} \\
2 & & & \\
2^{nR'} & & & \\
\end{array}
\]

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Non-causal cooperative binning

- Encoding constraints

\[ \tilde{R} > I(U; S) \]
\[ \tilde{R} < H(Z|X_r, U, S) \]
\[ R_U > \tilde{R} \]
Non-causal cooperative binning

- Encoding constraints

\[ \tilde{R} > I(U; S) \]
\[ \tilde{R} < H(Z|X_r, U, S) \]
\[ R_U > \tilde{R} \]

- Decoding constraints

\[ R' + \tilde{R} < H(Z|X_r, U, S) \]
\[ R' + \tilde{R} < R_U \]
\[ R'' < I(X; Y|Z, X_r, U, S) \]
\[ R < I(X, X_r; Y|S) \]
Fourier-Motzkin elimination

<table>
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<tr>
<th>Begin Elimination</th>
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<td></td>
<td>R'+Rtilde &lt; Ru</td>
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<tr>
<td></td>
<td>R'+Rtilde &lt; H(Z</td>
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<td>Rtilde &gt; I(U;S)</td>
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User Guide

Exit

http://www.ee.bgu.ac.il/~fme-it/
Non-causal cooperative binning

Capacity:

\[ C = \max \min \{ I(X, X_r; Y|S), I(X; Y|X_r, U, Z, S') + H(Z|X_r, U, S') - I(U; S) \} \]
Non-causal cooperative binning

- **Capacity:**
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  \]

- **Coding distribution:** \( P_{U|S} P_{X_r|U} P_{X|X_r, U, S} \).
Non-causal cooperative binning

- **Capacity:**

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\]

- **Coding distribution:**

\[
P_{U|S} P_{X_r|U} P_{X|X_r, U, S}.
\]

- **Distributions constraint**

\[
I(U; S) < H(Z|X_r, U, S')
\]
Non-causal cooperative binning

- **Capacity:**

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C = \max \min \left\{ I(X, X_r; Y|S), I(X; Y|X_r, U, Z, S') + H(Z|X_r, U, S') - I(U; S) \right\}
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- **Distributions constraint**

\[
I(U; S) < H(Z|X_r, U, S')
\]

- **Includes causal case / no-states !**
Without states [El-Gamal, Aref, 1982]:

\[ C = \max_{P_{X,X_r}} \min \{ I(X,X_r;Y), H(Z|X_r) + I(X;Y|X_r,Z) \} \]
Summary - semi-deterministic relay channels

- Without states [El-Gamal, Aref, 1982]:
  \[ C = \max_{P_{X, X_r}} \min \{ I(X, X_r; Y), H(Z|X_r) + I(X; Y|X_r, Z) \} \]

- Causal states [Kolte, Özgür, Permuter, 2016]:
  \[ C = \max_{P_{X_r} P_{X|X_r, S}} \min \{ I(X, X_r; Y|S), H(Z|S, X_r) + I(X; Y|S, X_r, Z) \} \]
Summary - semi-deterministic relay channels

- Without states [El-Gamal, Aref, 1982]:
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- Causal states [Kolte, Özgür, Permuter, 2016]:
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- Non-causal states [This work]:
  \[ C = \max \min \left\{ I(X, X_r; Y|S), I(X; Y|X_r, U, Z, S) + H(Z|X_r, U, S) - I(U; S) \right\} \]
  for \( P_{U|S} P_{X_r|U} P_{X|X_r,U,S} \) s.t. \( I(U; S) < H(Z|X_r, U, S) \).
Summary - semi-deterministic relay channels

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- Causal states [Kolte, Özgür, Permuter, 2016]:
  \[ C = \max_{P_{X_r}} \min_{P_{X|X_r}, S} \left\{ I(X, X_r; Y|S), H(Z|S, X_r) + I(X; Y|S, X_r, Z) \right\} \]

- Non-causal states [This work]:
  \[ C = \max \min \left\{ I(X, X_r; Y|S), I(X; Y|X_r, U, Z, S) + H(Z|X_r, U, S) - I(U; S) \right\} \]

for \( P_{U|S}P_{X_r|U}P_{X|X_r, U, S} \) s.t. \( I(U; S) < H(Z|X_r, U, S) \).

Thank you!

(FME software: http://www.ee.bgu.ac.il/~fme-it/)