Cooperative Binning for Semi-deterministic Channels with Non-causal State Information

Ido B. Gattegno  
Ben-Gurion University  
idobenja@post.bgu.ac.il

Haim H. Permuter  
Ben-Gurion University  
haimp@bgu.ac.il

Shlomo (Shitz) Shamai  
Technion  
sshlomo@ee.technion.ac.il

Ayfer Özgür  
Stanford  
aozgur@stanford.edu

Abstract—The capacity of two semi-deterministic channels with the presence of non-causal channel state information (CSI) is characterized. The first channel is a state-dependent semi-deterministic relay channel. The CSI is available only at the transmitter and receiver, but not at the relay. The second channel is a state-dependent multiple access channel (MAC) with partial cribbing and CSI only at one transmitter and the receiver.

In the semi-deterministic relay channel without states, the capacity can be achieved using partial-decode-forward scheme. The transmission is split to blocks; in each block, the relay decodes a part of the message and cooperation is established using those bits. When the channel depends on a state, the decoding procedure at the relay reduces the transmission rate. Recently, a cooperative bin forward scheme has been proposed which establishes cooperation without requiring the relay to decode a part of the message. In this scheme, the relay maps its received sequence, which is a deterministic function of the transmitted sequence, into bins. The transmitter coordinates its transmission with the bin index that is chosen by the relay. This scheme achieves the capacity when the CSI is available causally.

In this work, we present a variation of the cooperative-bin-forward scheme that achieves capacity for non-causal CSI. The bin index corresponding to the deterministic output of the relay is selected by the transmitter in such a way that the relay’s transmission is coordinated with the states. This coding scheme also applies for the MAC with partial cribbing and non-causal CSI at one transmitter and receiver. The capacity is achieved by the new variation of cooperative bin-forward. On top of that, we show an example in which the capacity with non-causal CSI is strictly greater than with causal CSI.

I. INTRODUCTION

Semi-deterministic models describe a variety of communication problems in which there exists a deterministic link between a transmitter and a receiver. This work focuses on the semi-deterministic relay channel (SD-RC) and the multiple access channel (MAC) with partial cribbing and CSI only at the encoder and decoder. The deterministic link between the transmitter and the relay or other the encoder may also be a function of the state. We characterize the capacity of these channels using a variation of a cooperative-bin-forward scheme that was suggested by Kolte et al. in [1].

The capacity of the relay channel was first studied by van der Muelen [2]. In the relay channel, an encoder receives a message, denoted by $M$, and sends it to a decoder over a channel with two outputs. A relay observes one of the channel outputs, denoted by $Z$, and uses past observations in order to help the encoder deliver the message. The decoder observes the other output, denoted by $Y$, and uses it to decode the message that was sent by the encoder. Cover and El-Gamal [3] established achievable rates for the general relay channel, using a partial-decode-forward scheme. If the channel is semi-deterministic (i.e. the output to the relay is a function of the channel inputs), El-Gamal and Aref [4] showed that this scheme achieves the capacity. Partial-decode-forward operates as follows: first, the transmission is divided into $B$ blocks, each of length $n$; in each block $b$ we send a message $M^{(b)}$, at rate $R$, that is independent of the messages in the other blocks. The message is split; after each transmission block, the relay decodes a part of the message and forwards it to the decoder in the next block using its transmission sequence. Since the encoder also knows the message, it can cooperate with the relay in the next block. However, when the channel depends on a state that is unknown to the relay, the partial-decode-forward scheme does not achieve the capacity [1]. The partial-decoding procedure at the relay is too restrictive since the relay is not aware of the channel state.

Focusing on state-dependent SD-RC, we consider two situations: when the CSI is available in a causal or a non-causal manner. State-dependent relay channels were studied in [1], [5]–[12]. Kolte et al. [1] derived the capacity of state-dependent SD-RC with causal CSI and introduced a cooperative-bin-forward coding scheme. The relay does not have to explicitly recover the message bits; instead, it agrees with the encoder on a map from the deterministic outputs space $Z^n$ to a bin index. This index is used by the relay to choose the next transmission sequence. Note that this cooperative-binning is independent of the state and, therefore, can be used by the relay. The encoder is also aware of this index (since the output is deterministic) and coordinates with the relay in the next block. However, the relay’s transmission does not depend on the state. The relay observes only strictly past outputs of the channel, which may contain information on past states but not on the current state.

We characterize the capacity of the SD-RC with non-causal CSI using a new variation of the cooperative-bin-forward scheme. The non-causal CSI allows dependency between the relay’s transmission and the state. The encoder can perform a look-ahead operation and transmit to the relay information about the upcoming states. The relay can still agree with
the encoder on a map, and in each transmission block, the encoder can choose carefully which index it causes the relay to see. The encoder chooses an index such that it reveals compressed state information to the relay, using an auxiliary cooperation codeword. This scheme can be used in other semi-deterministic models, such as the multiple access channel (MAC) with strictly causal partial cribbing and non-causal CSI.

The MAC with cooperation can also be viewed as a semi-deterministic model, due to the deterministic cooperation link. MAC with cribbing was first introduced by Willems [13], in which one transmitter has access to (is cribbing) the transmission of the other. A generalization of the cribbing is partial and controlled cribbing, introduced by Asnani and Permuter in [14]. The cribbed information is a deterministic function of the transmission sequence. When states are known causally at the first encoder (while the other is cribbing), Kolte et al. [1] derived the capacity, which is achieved by cooperative-bin-forward. The variation of the cooperative-bin-forward scheme achieves the capacity when the states are known non-causally.

II. PROBLEM DEFINITION AND MAIN RESULTS

A. Semi-Deterministic Relay Channel

We begin with a state dependent SD-RC, depicted in Fig. 1. This channel depends on a state $S_i \in \mathcal{S}$, which is known non-causally to the encoder and decoder, but not to the relay. An encoder sends a message $M$ to the decoder through a channel with two outputs. The relay observes an output $Z^n$ of the channel, which at time $i$ is a deterministic function of the channel inputs, $X_i$ and $X_{r,i}$, and the state (i.e., $Z_i = z(X_i, X_{r,i}, S_i)$). Based on past observations $Z^{i-1}$ the relay transmits $X_r$, in order to assist the encoder. The decoder uses the state information and the channel output $Y^n$ in order to estimate $M$. The channel is memoryless and characterized by the joint PMF $p_{Y,Z|X,X_r,S} = 1_{Z|X,X_r,S}p_{Y|Z,X,X_r,S}$.

Definition 1 (Code for SD-RC) A $(R,n)$ code $C_n$ for the SD-RC is defined by

\[
x^n : \{1 : 2^{nR}\} \times \mathcal{S}^n \to \mathcal{X}^n
\]

\[
x_{r,i} : Z^{i-1} \to \mathcal{X}_r
\]

\[
m^n : \mathcal{Y}^n \times \mathcal{S}^n \to \{1 : 2^{nR}\}
\]

B. Multiple Access Channel with Partial Cribbing

Consider a MAC with partial cribbing and non-causal state information, as depicted in Fig. 2. This channel depends on a state $(S_1, S_2)$ sequence that is known to the decoder, and each encoder $w \in \{1, 2\}$ has non-causal access to one state component $S_w \in \mathcal{S}_w$, Each encoder $w$ sends a message $M_w$ over the channel. Encoder 2 is cribbing Encoder 1; the cribbing is strictly causal, partial and controlled by $S_1$. Namely, the cribbed signal at time $i$, denoted by $Z_i$, is a deterministic function of $X_{1,i}$ and $S_{1,i}$.

Definition 2 (Achievable rate) A rate $R$ is achievable if there exists $(R,n)$ such that

\[
P_e(C_n) \triangleq \Pr[C_n[\hat{m}Y^n, S^n] \neq M] \leq \epsilon
\]

for any $\epsilon > 0$ and some sufficiently large $n$.

The capacity is defined to be the supremum of all achievable rates.

Theorem 1 The capacity of the SD-RC with non-causal CSI, depicted in Figure 7 is given by

\[
C = \max \min \{I(X;Z;Y|S), \ I(X;Y|X_r,Z,S,U) + H(Z|X_r,S,U) - I(U;S)\}
\]

where the maximum is over $p_U|S|p_{X,U}|X_{r,U}|S$ such that $I(U;S) \leq H(Z|X_r,S,U)$, where $Z = z(X,X_r, S)$ and $|U| \leq \min(|S|, |X_r|, |S||Y| + 1)$.

Let us investigate the capacity and the role of the auxiliary random variable $U$. The random variable $U$ is used to create empirical coordination between the encoder, the relay and the states, i.e., with high probability $(S^n, U^n, X^n)$ are jointly typical w.r.t. $p_{S,U,X}$. Note that the PMF factorizes as $p_{U|S|p_{X,U}|X_{r,U}|S}$ the random variable $X_r$, which represents the relay, depends on $S$ through the random variable $U$. This dependency represents the state knowledge at the relay, using an auxiliary codeword $U^n$.

B. Multiple Access Channel with Partial Cribbing

Consider a MAC with partial cribbing and non-causal state information, as depicted in Figure 2. This channel depends on a state $(S_1, S_2)$ sequence that is known to the decoder, and each encoder $w \in \{1, 2\}$ has non-causal access to one state component $S_w \in \mathcal{S}_w$. Each encoder $w$ sends a message $M_w$ over the channel. Encoder 2 is cribbing Encoder 1; the cribbing is strictly causal, partial and controlled by $S_1$. Namely, the cribbed signal at time $i$, denoted by $Z_i$, is a deterministic function of $X_{1,i}$ and $S_{1,i}$.

Definition 3 (Code for MAC) A $(R_1, R_2, n)$ code $C_n$ for the state-dependent MAC with strictly causal partial cribbing and
two state components is defined by
\[ x^n_i : [1 : 2^{nR_1}] \times S^n_1 \to X^n_i \]
\[ x_{2,i} : [1 : 2^{nR_2}] \times S^n_2 \times z^{i-1} \to X^n_2 \quad 1 \leq i \leq n \]
\[ \tilde{m}_1 : Y^n \times S^n_1 \times S^n_2 \to [1 : 2^{nR_1}] \]
\[ \tilde{m}_2 : Y^n \times S^n_1 \times S^n_2 \to [1 : 2^{nR_2}] \]
for any \( \epsilon > 0 \) and some sufficiently large \( n \).

**Definition 4 (Achievable rate-pair)** A rate-pair \((R_1, R_2)\) is achievable if there exists a code \(C_n\) such that
\[ P_e(C_n) = \Pr_{C_n}[(M_1, M_2) \neq (M_1, M_2)] \leq \epsilon \]
for any \( \epsilon > 0 \) and some sufficiently large \( n \).

The capacity region of this channel is defined to be the union of all achievable rate-pairs.

**Theorem 2** The capacity region for discrete memoryless MAC with non-causal CSI and strictly causal cribbing in Fig. 4 is given by the set of rate pairs \((R_1, R_2)\) that satisfy
\[ R_1 \leq I(X_1; Y|X_2, Z, S_1, S_2, U) + H(Z|S_1, U) \quad (3a) \]
\[ R_2 \leq I(X_2; Y|X_1, S_1, S_2) \quad (3b) \]
\[ R_1 + R_2 \leq I(X_1, X_2; Y|Z, S_1, S_2, U) + H(Z|S_1, U) \quad (3c) \]
\[ R_1 + R_2 \leq I(X_1, X_2; Y|S_1, S_2) \quad (3d) \]
for PMFs of the form \(p_{X_1, U|S_1} p_{X_2,U|S_2} \) with \( Z = z(X_1, S_1) \), that satisfies
\[ I(U; S_1|S_2) \leq H(Z|S_1, U), \quad (3e) \]
and \( |U| \leq \min \{|S_2||S_1|X_1|X_2| + 2, |S_1||S_2||Y| + 3\} \).

We note here that when \( S_2 \) is degenerated, i.e., there is only one state component, the capacity region is given by degenerating \( S_2 \) in the information terms.

Here, the auxiliary random variable \( U \) plays a double role. The first role is similar to the role in the SD-RC; it creates dependency between \( X_2 \) and \( S_1 \). This is done using a cooperation codeword \( U^n \); Encoder 1 selects a codeword that is coordinated with the states. Encoder 2 uses this codeword in order to cooperate. Since the codeword depends on the state, so does \( X^n_2 \). When there are two state components, the second component is used by Encoder 2 to select the cooperation codeword from a collection. The second role is to generate a common message between the encoders.

### III. NON-CAUSAL COOPERATIVE BINNING

We will now provide the outline for achieving the capacity of the SD-RC in Theorem 1. Proofs will be given in [13].

**Lemma 1 (Indirect covering lemma)** Let \( \{Z^n(k)\}_{k \in [1 : 2^n]} \) be a collection of sequences, each sequence is drawn i.i.d according to \( \prod_{i=1}^n P_{Z|V}(z_i|v_i) \). For every \( z^n \in Z^n \), let Bin\((z^n) \sim Unif[1 : 2^{nR_v}]\). For any \( \delta_1, \delta_2 > 0 \), if
\[ R < H(Z|V) - \delta_1 \]
\[ R < R_B - \delta_2, \]
then,
\[ \lim_{n \to \infty} \Pr \left[ \left| \{i : \exists z(k) \text{ s.t. Bin}(z^n(k)) = i \} \right| < 2^{n(R - \Delta_n)} \right] = 0 \]
where \( \Delta_n \to 0 \) as \( n \to \infty \).

In block-Markov, the transmission is divided to \( B \) blocks. We denote \( b \) in the superscript for the index of the block. In each block, the transmitter sends a message \( m(b), b \in [1 : B] \). The achieved rate is the average transmission rate, which is \( R = \frac{1}{B} \sum_{b=1}^B R(b) \), where \( R(b) \) is the rate in block \( b \in [1 : B] \). The block rates \( R(b) \) are set to \( R \), except \( R(1) = R(B) = 0 \). Respectively, \( m(b) \) is split to \( (m(l)(b), m(m)(b)) \) with corresponding rates \( (R', R'') \).

The cooperation between the encoder and the relay is achieved by using a superposition code on a cooperation codeword. In order to agree on the same cooperation codeword, the deterministic output space \( Z^n \) is partitioned into bins. This map is a part of the codebook and known to both the encoder and the relay before the transmission. Since the link from the encoder to the relay is deterministic, the encoder can determine what sequence the relay will observe during the block. Consequently, they both can agree on the same index. This index has two roles: 1) establish coordination between the encoder’s transmission and the relay’s transmission, and 2) reveal a compressed version of the state sequence of the next block.

![Fig. 3: Cooperative binning - choosing cooperation codeword according to states.](image-url)

**Codebook:** Let Bin\((z^n) \sim Unif[1 : 2^{nR_v}]\), \( \forall z^n \in Z^n \). For each bin index \( l \), draw a auxiliary codeword \( u^n(l) \sim \prod_{i=1}^n p_{U|V}(u_i|v_i) \), \( l \in [1 : 2^{nR_v}] \), where \( p_{U|V}(u|v) = \sum_{s \in S} p_{V|S}(v|s)p(s) \). This collection of codewords is used for superposition coding. For each \( m(b) \), draw a collection of \( 2^{nR_v} \) codewords \( z^n \) according to \( p_{Z|X, U, S} \). Each
The advantage of sliding window is that the decoder do not establish and, the relay’s codeword is coded, as depicted in Fig. 4. Case (a) is a causal case, and (b) is a non-causal case. Respectively, the channels in Fig. 4a and 4b are special cases of the causal and non-causal state dependent MAC with partial cribbing.

To apply the MAC with partial cribbing to this case, consider the following situation with only one state component. Encoder 1 has no access to the channel, i.e., \( p_Y|X_1, X_2 = p_Y|X_2 \), and no message to send (\( H_1 = 0 \)). Its only job is to assist Encoder 2 by compressing the CSI and sending it via a private link, which is the partial cribbing with \( z(x_1, s) = x_1 \).

Since this is a point-to-point (P2P) setup, it is a bit surprising that the non-causal CSI increases capacity; when the states are perfectly provided to the encoder, the capacity with causal CSI and with non-causal CSI coincide. However, when there is a state-encoder, the alphabet size of \( X_1 \) can enforce lossy quantization on the state in the causal case, while in the non-causal case, the states can be losslessly compressed. For every channel \( p_Y|X_2,S \) and states distribution \( p_S \),

\[
C_{nc} = \max_{p_{U}|p_{X_2}|S} I(X_2;Y|S,U) \quad (8a)
\]

\[
C_{c} = \max_{1_{X_1:Y|P_{X_2}|S}} I(X_2;Y|S,X_1) \quad (8b)
\]

where \( C_{nc} \) and \( C_c \) are the capacity of non-causal and causal CSI configurations, respectively. Note that the difference be-
between the capacity characterizations is the auxiliary random variable $U$. The idea is that when the CSI is known non-causally we can compress $S^n$ using an auxiliary codeword $U^n$, while in a causal case we cannot.

Assume that the states distribution is

$$p_S(s) = \begin{cases} \frac{b}{2} & \text{if } s = 0, 1 \\ 1 - p & \text{if } s = 2. \end{cases}$$

(9)

For each state there is a different channel; these channels are depicted in Fig. 5, a Z-channel for $s = 0$, an S-channel for $s = 1$, where both share the same parameter $\alpha$, and a noiseless channel for $s = 2$.

Assume that $X_1$ is binary, and $p$ is small enough, for instance $p = 0.2$, such that

$$H(S) < \log_2 |X_1| = 1.$$  

(10)

Therefore, taking $U = S$ satisfies $I(U; S) = H(S) \leq 1$ and results in the non-causal capacity

$$C_{nc} = \frac{p}{2} (C_{Z, \text{channel}}(\alpha) + C_{S, \text{channel}}(\alpha)) + (1 - p)$$  

(11)

where

$$C_{Z, \text{channel}}(\alpha) = C_{S, \text{channel}}(\alpha)$$  

$$= H_b \left( \frac{2 H_b(\alpha)/\alpha}{1 + 2 H_b(\alpha)/\alpha} \right) - \frac{H_b(\alpha)/\alpha}{1 + 2 H_b(\alpha)/\alpha}$$

(12)

and $H_b(\cdot)$ is the binary entropy. On the other hand, the capacity for causal CSI is

$$C_c = \max_\beta \left[ \frac{p}{2} (C_{Z, \text{channel}}(\alpha) + H_b(\beta + \beta \alpha - \beta H_b(\alpha)) + (1 - p)H_b(\beta) \right].$$

(13)

The capacity can be achieve by one of several deterministic functions $x_{1, i}(S^i)$. Each function, maps both $S = 2$ and $S = 0$ to one letter, and $S = 0$ to the other letter, respectively. Note that this operation causes a lossy quantization of the CSI. For comparison, we also provide the capacity when there is no CSI at the encoder, which is

$$C_{nc, \text{CSI}} = p \left( H_b \left( \frac{1 + \alpha}{2} \right) - 0.5 H_b(\alpha) \right) + (1 - p).$$

(14)

The capacity of the channels (non-causal, causal, no CSI) for $p = 0.2$ are summarized in Table I. There are two points where the three channels result in the same capacity. The first is when $\alpha = 0$; in this case, the channel is noiseless for $s = 0, 1, 2$ and the capacity is 1. There is no need for CSI at the encoder and, therefore, the capacity is the same (among the three cases). The second point is when $\alpha = 1$; the channel is stuck at 0 and stuck at 1 for $s = 0$ and $s = 1$, respectively, and noiseless for $s = 2$. In this case we can set $P_{X_1|S}(1|s) = 0.5$ for every $s \in S$ and achieve the capacity. Therefore, the encoder does not use the CSI in these cases. However, for every $\alpha \in (0, 1)$, the capacity of the non-causal case is strictly greater than of the others, which confirms that non-causal CSI indeed increases the capacity region.

V. ACKNOWLEDGMENTS

This work was supported by the Heron Consortium via the Minister of Economy and Science.

REFERENCES


<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>No-CSI</th>
<th>Causal CSI</th>
<th>Non-causal CSI</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.5</td>
<td>0.8623</td>
<td>0.8633</td>
<td>0.8644</td>
</tr>
<tr>
<td>1</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
</tr>
</tbody>
</table>