The Capacity of the Trapdoor Channel with Feedback

Haim Permuter

Based on work with

Paul Cuff, Benjamin Van Roy and Tsachy Weissman

Stanford University
Main Results of the Talk

1. capacity of the trapdoor channel with feedback

2. simple scheme that achieves feedback capacity
The trapdoor channel

Input: 1 0 1

Channel:

Output:

\[ s_t = s_{t-1} \oplus x_t \oplus y_t \]

\[ s_0 = 0 \]
The trapdoor channel

\[ s_t = s_{t-1} \oplus x_t \oplus y_t \]

\[ s_0 = 0, \]

\[ x_1 = 1 \]
The trapdoor channel

\[ s_t = s_{t-1} \oplus x_t \oplus y_t \]

\[ s_0 = 0, \]
\[ x_1 = 1, \quad s_1 = 1, \quad y_1 = 0, \]

Input \[ \begin{array}{c}
1 \\
0
\end{array} \]

Channel

Output \[ 0 \]
The trapdoor channel

\[ s_t = s_{t-1} \oplus x_t \oplus y_t \]

\begin{align*}
  s_0 &= 0, \\
  x_1 &= 1, \quad s_1 = 1, \quad y_1 = 0, \\
  x_2 &= 0,
\end{align*}
The trapdoor channel

\[ s_t = s_{t-1} \oplus x_t \oplus y_t \]

- \( s_0 = 0 \)
- \( x_1 = 1, \ s_1 = 1, \ y_1 = 0, \)
- \( x_2 = 0, \ s_2 = 1, \ y_2 = 0, \)
The trapdoor channel

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\[ x_1 = 1, \ s_1 = 1, \ y_1 = 0, \]
\[ x_2 = 0, \ s_2 = 1, \ y_2 = 0, \]
\[ x_3 = 1, \]
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The trapdoor channel

Introduced by David Blackwell in 1961. [Ash65], [Ahlsvede & Kaspi 87], [Ahlsvede 98], [Kobayashi 02].

Another appropriate name for this channel is chemical channel.
Communication setting

Finite State Channel (FSC) property: \( p(y_i, s_i | x^i, s^{i-1}, y^{i-1}) = p(y_i, s_i | x_i, s_{i-1}) \)

Unifilar channel [Ziv85]: \( s_t = f(s_{t-1}, x_t, y_t) \)

Figure 1: Unifilar FSC with feedback
Main ingredients

1. Directed information.
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2. Dynamic program average-reward.
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2. Dynamic program average-reward.

3. Value iteration.
Main ingredients

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4. Bellman equation.
Main ingredients

1. Directed information.

2. Dynamic program average-reward.

3. Value iteration.

4. Bellman equation.

5. Homework question given by Tom Cover.
Feedback capacity of FSC

Lower and Upper bound

\[ C_{FB} \geq \lim_{N \to \infty} \frac{1}{N} \max_{\{p(x_i|x_{i-1},y_{i-1})\}_{i=1}^{N}} \min_{s_0} I(X^N \to Y^N|s_0) \]

\[ C_{FB} \leq \lim_{N \to \infty} \frac{1}{N} \max_{\{p(x_i|x_{i-1},y_{i-1})\}_{i=1}^{N}} \max_{s_0} I(X^N \to Y^N|s_0) \]

[Permuter, Weissman and Goldsmith ISIT06]

where

\[ I(X^n \to Y^n) \triangleq \sum_{i=1}^{n} I(X^i;Y_i|Y^{i-1}) \]
Feedback capacity of FSC

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[Permuter, Weissman and Goldsmith ISIT06]

where

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\]

In the trapdoor channel any state \(s_t\) can be reached from any state \(s_{t-1}\) with positive probability and hence we get

\[
C_{FB} = \lim_{N \to \infty} \frac{1}{N} \max_{\{p(x_i|x^{i-1},y^{i-1})\}_{i=1}^N} I(X^N \to Y^N)
\]
Directed information was defined by Massey in 1990,

$$I(X^n \to Y^n) \equiv \sum_{i=1}^{n} I(X^i; Y_i | Y^{i-1})$$
Directed information

Directed Information was defined by Massey in 1990,

\[ I(X^n \rightarrow Y^n) \triangleq \sum_{i=1}^{n} I(X^i; Y_i|Y^{i-1}) \]

\[ I(X^n; Y^n) = \sum_{i=1}^{n} I(X^n; Y_i|Y^{i-1}) \]
Directed information - intuition

If there is no feedback

\[ I(X^n; Y^n) = I(X^n \to Y^n) \]
Directed information - intuition

If there is no feedback

\[ I(X^n; Y^n) = I(X^n \rightarrow Y^n) \]

Perfect feedback

\[ I(X^n; Y^n) = I(X^n \rightarrow Y^n) + I(Y^{n-1} \rightarrow X^n) \]
Directed information - intuition

If there is no feedback

\[ I(X^n; Y^n) = I(X^n \rightarrow Y^n) \]

Perfect feedback

\[ I(X^n; Y^n) = I(X^n \rightarrow Y^n) + I(Y^{n-1} \rightarrow X^n) \]

Deterministic feedback \( k_i(y_i) \)

\[ I(X^n; Y^n) = I(X^n \rightarrow Y^n) + I(K^{n-1} \rightarrow X^n) \]
Feedback capacity

\[ C_{FB} = \lim_{N \to \infty} \frac{1}{N} \max_{\{p(x_t|x^{t-1},y^{t-1})\}^N_{t=1}} I(X^N \to Y^N) \]
Feedback capacity

\[ C_{FB} = \lim_{N \to \infty} \frac{1}{N} \max_{\{p(x_t|x^{t-1},y^{t-1})\}_{t=1}^N} \sum_{t=1}^N I(X^N \to Y^N) \]

\[ = \lim_{N \to \infty} \frac{1}{N} \max_{\{p(x_t|x^{t-1},y^{t-1})\}_{t=1}^N} \sum_{t=1}^N I(X^t;Y_t|Y^{t-1}) \]
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Feedback capacity

\[ C_{FB} = \lim_{N \to \infty} \frac{1}{N} \max_{t=1}^{N} \{p(x_t|x_{t-1},y_{t-1})\} \sum_{t=1}^{N} I(X^t; Y^t_{t-1}) \]

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\[ = \sup_{\{p(x_t|s_{t-1},y_{t-1})\}_{t\geq1}} \liminf_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} I(X_t; S_{t-1}; Y_t|Y^t_{t-1}) \]
Feedback capacity and dynamic programming (DP)

DP consists of states $\beta_{t-1}$, actions $u_t(\beta_{t-1})$, and disturbance $w_t$.

state:

$$\beta_{t-1} = p(s_{t-1}|y^{t-1}), \quad \beta \in [0, 1]$$

action:

$$u_t = p(x_t|s_{t-1}), \quad u_t \in [0, 1] \times [0, 1]$$

disturbance:

$$w_t = y_{t-1},$$

$$\beta_t = F(\beta_{t-1}, u_t, w_t), \quad t = 1, 2, 3, \ldots,$$

reward function per unit time

$$g(\beta_{t-1}, u_t) = I(X_t, S_{t-1}; Y_t|\beta_{t-1}).$$

[Tatikonda00], [Yang, Kavčić and Tatikonda05]
Dynamic programing operator, $T$

The dynamic programming operator $T$ is given by

$$(TJ)(\beta) = \sup_{u \in U} \left( g(\beta, u) + \int P_w(dw|\beta, u) J(F(\beta, u, w)) \right)$$
Dynamic programming operator, $T$

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$$(TJ)(\beta) = \sup_{u \in U} \left( g(\beta, u) + \int P_w(dw|\beta, u) J(F(\beta, u, w)) \right)$$

$$
(TJ)(\beta) = \sup_{0 \leq \delta \leq \beta, 0 \leq \gamma \leq 1-\beta} \left( H \left( \frac{1}{2} + \frac{\delta - \gamma}{2} \right) + \delta + \gamma - 1 + \frac{1 + \delta - \gamma}{2} J \left( \frac{2\delta}{1 + \delta - \gamma} \right) \right) \\
+ \frac{1 - \delta + \gamma}{2} J \left( 1 - \frac{2\gamma}{1 - \delta + \gamma} \right) 
$$

Properties

- Preservation of *concavity*: if $J$ is concave then $TJ$ is concave.
- Preservation of *continuity*: if $J$ is continuous then $TJ$ is continuous.
- Preservation of *symmetry*: if $J$ is symmetric then $TJ$ is symmetric.
Computational study

Executed 20 value iterations: \( J_{k+1}(\beta) = (T J_k)(\beta) \)
Computational study

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HW question from Prof. Cover class

*Entropy rate.* Find the maximum entropy rate of the following two-state Markov chain:

![Diagram of a two-state Markov chain with states A and B, transition probabilities p and 1-p.](attachment:image.png)
Computational study

Executed 20 value iterations: \[ J_{k+1}(\beta) = (TJ_k)(\beta) \]

\[ C_{FB} \approx 0.694 \]

HW question from Prof. Cover class

**Entropy rate.** Find the maximum entropy rate of the following two-state Markov chain:

![Diagram of a two-state Markov chain with transitions labeled 1-p and p from A to B, and 1 from B to A.]

**Solution:** The entropy rate is \( \log_2 \phi = 0.6942... \), where \( \phi \) is the golden ratio: \( \phi = \frac{\sqrt{5}+1}{2} \).
20th Value iteration

\[ J_{20} \]

\[ \delta, \; \text{iteration}=20 \]

\[ \gamma, \; \text{iteration}=20 \]

histogram of beta

relative frequency
Conjecture of the solution to Bellman equation

\[ H(\beta) - \rho \beta + c_2 \quad \text{and} \quad H(\beta) + \rho \beta + c_1 \]

\[ \frac{\sqrt{5} - 1}{2} \beta + \frac{3 - \sqrt{5}}{2} \]

\[ \frac{3 - \sqrt{5}}{2} \]
Bellman equation

Theorem 1. If there exists \((J(\beta), \rho)\) that satisfies

\[ J(\beta) = (TJ)(\beta) - \rho, \]

then \(\rho\) is the optimal average reward.
Verifying our conjecture

Construct value iteration function $J_k(\beta)$ as follows. Let $J_0(\beta)$ be the pointwise maximum among concave functions satisfying $J_0(\beta) = \tilde{J}(\beta)$ for $\beta \in [b_1, b_4]$

$$J_{k+1}(\beta) = (TJ_k)(\beta) - \tilde{\rho},$$

- concave, continuous and symmetric
- fixed point: for $\beta \in [b_1, b_4]$ , $J_k(\beta) = \tilde{J}(\beta)$
- monotonically nonincreasing in $k$
- converges uniformly to $J^*(\beta)$

Since the sequence $J_{k+1} = TJ_k - \tilde{\rho}1$ converges uniformly and $T$ is sup-norm continuous, $J^* = TJ^* - \tilde{\rho}1$. 
A scheme that achieves capacity

Question

*Number of sequences.* To first order in the exponent, what is the number of binary sequences of length $n$ with no two consecutive 1’s?

0010101010010101...
A scheme that achieves capacity

Question

Number of sequences. To first order in the exponent, what is the number of binary sequences of length $n$ with no two consecutive 1’s?

00101010100101...

Solution The number of sequences of length $n$ with this property, is the $n^{th}$ Fibonacci number, $f_n \doteq \phi^n$.

The scheme

Let us denote such a sequence by $r^n$. Map each message $m$ to a sequence $[r^n(m)]$.

encoder: $x_t = s_{t-1} \oplus r_t, t = 1, ..., n$ and $x_{n+1} = s_n$.

decoder: The decoder can decode this sequence error-free!
Conclusions

• The capacity of the trapdoor channel with feedback is

\[ C_{FB} = \log_2 \phi, \]

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- Closed form solution to an infinite horizon average-reward dynamic.
Conclusions

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where $\phi$ is the golden ratio.

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Thank You!