A new coding scheme for cooperation in semi-deterministic channels

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Communication and Information Theory Colloquium
Technion
Aug 2015
Uplink Communication
Uplink Communication

cribbing
Uplink Communication

Channel state $S_1$

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Channel state $S_2$
The communication setting considered in the talk

Encoder 1

Encoder 2

MAC

Decoder

$P_{Y|X_1,X_2}$
The communication setting considered in the talk

Perfect Cribbing [Willem82]

Encoder 1

Encoder 2

MAC

Decoder

\[ P_{Y|X_1,X_2} \]

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The communication setting considered in the talk

Partial (deterministic-function) cribbing [P./Asnani13]

\[
\begin{align*}
\text{Encoder 1} & \quad X_{1,i}(m_1, Z_{2}^{i-1}) \\
\text{Encoder 2} & \quad X_{2,i}(m_2, Z_{1}^{i-1}) \\
\end{align*}
\]

\[
\begin{align*}
Z_{1,i} = g_1(X_{1,i}) \\
Z_{2,i} = g_2(X_{2,i}) \\
\end{align*}
\]

\[
\begin{align*}
P_{Y|X_1,X_2} \\
\end{align*}
\]

\[
\begin{align*}
\hat{m}_1(Y^n) \\
\hat{m}_2(Y^n) \\
\end{align*}
\]

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The communication setting considered in the talk

Partial (deterministic-function) cribbing with state [Kolte/Özgür/P.15]

$P_{Y|X_1, X_2, S_1, S_2}$

$Z_{1,i} = g_1(X_{1,i})$

$Z_{2,i} = g_2(X_{2,i})$

$S_1, S_2 \sim P(s_1, s_2)$
Comments on MAC with cribbing

- Causal cribbing: $X_{1,i}(m_1, Z_2^{i})$
- Non-causal partial cribbing, $X_{1,i}(m_1, Z_2^{n})$, is open.
- Noisy cribbing [Bross/Steinberg/Tinguely10] is open.
- Perfect cribbing for Gaussian [Willems05] is trivial.
Capacity Region

**Theorem**

*Strictly causal:*

\[
\mathcal{R} = \begin{cases} 
R_1 \leq I(X_1; Y|X_2, Z_1, U) + H(Z_1|U), \\
R_2 \leq I(X_2; Y|X_1, Z_2, U) + H(Z_2|U), \\
R_1 + R_2 \leq I(X_1, X_2; Y|U, Z_1, Z_2) + H(Z_1, Z_2|U), \\
R_1 + R_2 \leq I(X_1, X_2; Y), \quad \text{for} \\
P(u)P(x_1, z_1|u)P(x_2, z_2|u)P(y|x_1, x_2). 
\end{cases}
\]

[P./Asnani13]
**Theorem**

**Strictly causal:**

\[ \mathcal{R} = \begin{cases} 
R_1 & \leq I(X_1; Y \mid X_2, Z_1, U) + H(Z_1 \mid U), \\
R_2 & \leq I(X_2; Y \mid X_1, Z_2, U) + H(Z_2 \mid U), \\
R_1 + R_2 & \leq I(X_1, X_2; Y \mid U, Z_1, Z_2) + H(Z_1, Z_2 \mid U), \\
R_1 + R_2 & \leq I(X_1, X_2; Y), \text{ for} \\
P(u)P(x_1, z_1 \mid u)P(x_2, z_2 \mid u)P(y \mid x_1, x_2). 
\end{cases} \]

**Causal:** The same but

\[ R_2 \leq I(X_2; Y \mid X_1, Z_2, U) + H(Z_2 \mid Z_1, U), \]

\[ P(u)P(x_1, z_1 \mid u)P(x_2, z_2 \mid z_1, u)P(y \mid x_1, x_2). \]
Theorem

**Strictly causal:**

\[
\mathcal{R} = \begin{cases} 
R_1 & \leq I(X_1; Y|X_2, Z_1, U) + H(Z_1|U), \\
R_2 & \leq I(X_2; Y|X_1, Z_2, U) + H(Z_2|U), \\
R_1 + R_2 & \leq I(X_1, X_2; Y|U, Z_1, Z_2) + H(Z_1, Z_2|U), \\
R_1 + R_2 & \leq I(X_1, X_2; Y), \text{ for} \\
P(u)P(x_1, z_1|u)P(x_2, z_2|u)P(y|x_1, x_2). 
\end{cases}
\]

**Causal:** The same but

\[
R_2 \leq I(X_2; Y|X_1, Z_2, U) + H(Z_2|Z_1, U), \\
P(u)P(x_1, z_1|u)P(x_2, z_2|z_1, u)P(y|x_1, x_2). 
\]

Two achievabilities:

- Partial decoding

---

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Theorem

Strictly causal:

\[ \mathcal{R} = \begin{cases} 
R_1 \leq I(X_1; Y|X_2, Z_1, U) + H(Z_1|U), \\
R_2 \leq I(X_2; Y|X_1, Z_2, U) + H(Z_2|U), \\
R_1 + R_2 \leq I(X_1, X_2; Y|U, Z_1, Z_2) + H(Z_1, Z_2|U), \\
R_1 + R_2 \leq I(X_1, X_2; Y), \text{ for} \\
P(u)P(x_1, z_1|u)P(x_2, z_2|u)P(y|x_1, x_2). 
\end{cases} \]

Causal: The same but

\[ R_2 \leq I(X_2; Y|X_1, Z_2, U) + H(Z_2|Z_1, U), \\
P(u)P(x_1, z_1|u)P(x_2, z_2|z_1, u)P(y|x_1, x_2). \]

Two achievabilities:

1. Partial decoding
2. Cooperative binning
Achievability proof: Partial decoding

- Split $R_1$ into $R_1 = R'_1 + R''_1$. 
Achievability proof: Partial decoding

- Split $R_1$ into $R_1 = R'_1 + R''_1$.
- Divide block of length $Bn$ into $B$ blocks of length $n$. 
Achievability proof: Partial decoding

- Split $R_1$ into $R_1 = R'_1 + R''_1$.
- Divide block of length $Bn$ into $B$ blocks of length $n$.
- Generate $2^{n(R'_1 + R'_2)}$ codewords $u^n$ i.i.d. $\sim P(u)$.
Achievability proof: Partial decoding

- Split $R_1$ into $R_1 = R'_1 + R''_1$.
- Divide block of length $Bn$ into $B$ blocks of length $n$.
- Generate $2^{n(R'_1 + R'_2)}$ codewords $u^n$ i.i.d. $\sim P(u)$.
- For each $u^n$, generate $2^{nR'_1}$ codewords $z^n_1$ according to i.i.d. $\sim P(z_1|u)$ and $2^{nR''_1}$ codewords $x^n_1 \sim P(x_1|z_1, u)$.
Achievability proof: Partial decoding

- Split $R_1$ into $R_1 = R'_1 + R''_1$.
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- Generate $2^{n(R'_1 + R'_2)}$ codewords $u^n$ i.i.d. $\sim P(u)$.
- For each $u^n$, generate $2^{nR'_1}$ codewords $z^n_1$ according to i.i.d. $\sim P(z_1 | u)$ and $2^{nR''_1}$ codewords $x^n_1 \sim P(x_1 | z_1, u)$
- Encoder 2 decodes at the end of block $b$ the message $m'_{1,b}$ from $z^n_1$. 
Achievability proof: Partial decoding

- Split $R_1$ into $R_1 = R'_1 + R''_1$.
- Divide block of length $Bn$ into $B$ blocks of length $n$.
- Generate $2^{n(R'_1 + R'_2)}$ codewords $u^n$ i.i.d. $\sim P(u)$.
- For each $u^n$, generate $2^{nR'_1}$ codewords $z^n_1$ according to i.i.d. $\sim P(z_1|u)$ and $2^{nR''_1}$ codewords $x^n_1 \sim P(x_1|z_1, u)$.
- Encoder 2 decodes at the end of block $b$ the message $m'_{1,b}$ from $z^n_1$. Hence $R'_1 \leq H(Z_1|U)$.
Achievability proof: Partial decoding

- Split $R_1$ into $R_1 = R'_1 + R''_1$.
- Divide block of length $Bn$ into $B$ blocks of length $n$.
- Generate $2^{n(R'_1 + R'_2)}$ codewords $u^n$ i.i.d. $\sim P(u)$.
- For each $u^n$, generate $2^{nR'_1}$ codewords $z^n_1$ according to i.i.d. $\sim P(z_1|u)$ and $2^{nR''_1}$ codewords $x^n_1 \sim P(x_1|z_1, u)$
- Encoder 2 decodes at the end of block $b$ the message $m'_{1,b}$ from $z^n_1$. Hence $R'_1 \leq H(Z_1|U)$.
- Block Markov code:
  $x^n_1$ is determined by $(m'_{1,b}, m''_{1,b})$ conditioned on $(m'_{1,b-1}, m'_{2,b-1})$. 

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Split $R_1$ into $R_1 = R'_1 + R''_1$.

Divide block of length $Bn$ into $B$ blocks of length $n$.

Generate $2^{n(R'_1 + R'_2)}$ codewords $u^n$ i.i.d. $\sim P(u)$.

For each $u^n$, generate $2^{nR'_1}$ codewords $z^n_1$ according to i.i.d. $\sim P(z_1|u)$ and $2^{nR''_1}$ codewords $x^n_1 \sim P(x_1|z_1, u)$.

Encoder 2 decodes at the end of block $b$ the message $m'_{1,b}$ from $z^n_1$. Hence $R'_1 \leq H(Z_1|U)$.

Block Markov code:
$x^n_1$ is determined by $(m'_{1,b}, m''_{1,b})$ conditioned on $(m'_{1,b-1}, m'_{2,b-1})$.

Backward decoding: At block $b$, we assume that $(m'_{1,b}, m'_{2,b})$ is known. Decode $m'_{1,b-1}, m'_{2,b-1}, m''_{2,b}$ and $m''_{1,b}$ from $Y^n$ using joint typicality.
After error analysis we obtain

\[ \begin{align*}
R'_1 & \leq H(Z_1|U), \\
R'_2 & \leq H(Z_2|U), \\
R''_1 & \leq I(X_1; Y|X_2, Z_1, U), \\
R''_2 & \leq I(X_2; Y|X_1, Z_2, U), \\
R''_1 + R''_2 & \leq I(X_1, X_2; Y|Z_1, Z_2, U), \\
R_1 + R_2 & \leq I(X_2, X_1; Y),
\end{align*} \]
After error analysis we obtain

\begin{align*}
R'_1 & \leq H(Z_1|U), \\
R'_2 & \leq H(Z_2|U), \\
R''_1 & \leq I(X_1; Y|X_2, Z_1, U), \\
R''_2 & \leq I(X_2; Y|X_1, Z_2, U), \\
R''_1 + R''_2 & \leq I(X_1, X_2; Y|Z_1, Z_2, U), \\
R_1 + R_2 & \leq I(X_2, X_1; Y),
\end{align*}

Using Fourier–Motzkin elimination we obtain the region.
Bin all the typical set $\mathcal{T}_\epsilon^{(n)}(Z_1)$ into $2^{nH(Z_1|U)+\delta}$. 
Bin all the typical set $\mathcal{T}_\epsilon^{(n)}(Z_1)$ into $2^{nH(Z_1|U)+\delta}$. 
For each bin $l$ generate codeword $u^n$ i.i.d. $\sim P(u)$. 

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Split $R_1$ into $R_1 = R'_1 + R''_1$.

Divide block of length $Bn$ into $B$ blocks of length $n$. 
Achievability proof: Cooperative Binning

- Bin all the typical set $\mathcal{T}_\epsilon^{(n)}(Z_1)$ into $2^{nH(Z_1|U)+\delta}$.
- For each bin $l$ generate codeword $u^n$ i.i.d. $\sim P(u)$.
- Split $R_1$ into $R_1 = R'_1 + R''_1$.
- Divide block of length $Bn$ into $B$ blocks of length $n$.
- For each $u^n$, generate $2^{nR'_1}$ codewords $z^n_1$ according to $\sim P(z_1|u)$ and $2^{nR''}$ codewords $X^n_1 \sim P(x_1|z_1,u)$.
Achievability proof: Cooperative Binning [Kolte/Özgür/P.15]

- Bin all the typical set $\mathcal{T}^ε(n)(Z_1)$ into $2^{nH(Z_1|U)+\delta}$.
- For each bin $l$ generate codeword $u^n$ i.i.d. $\sim P(u)$.
- Split $R_1$ into $R_1 = R'_1 + R''_1$.
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- Encoder 2 finds the associate bin from $Z^n_1$ and cooperatively transmits $x^n_2(m_{2,b},u^n)$.
Achievability proof: Cooperative Binning [Kolte/Özgür/P.15]

- Bin all the typical set $\mathcal{T}_\epsilon^{(n)}(Z_1)$ into $2^{nH(Z_1|U)+\delta}$.
- For each bin $l$ generate codeword $u^n$ i.i.d. $\sim P(u)$.
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- Encoder 2 finds the associate bin from $Z^n_1$ and cooperatively transmits $x^n_2(m_2,b,u^n)$.
- $X^n_1$ is determined by $(m'_1,b,m''_1,b)$ conditioned on $u^n$. 
Bin all the typical set $\mathcal{T}^{(n)}(Z_1)$ into $2^n H(Z_1 | U) + \delta$.

For each bin $l$ generate codeword $u^n$ i.i.d. $\sim P(u)$.

Split $R_1$ into $R_1 = R'_1 + R''_1$.

Divide block of length $Bn$ into $B$ blocks of length $n$.

For each $u^n$, generate $2^{nR'_1}$ codewords $z^n_1$ according to $\sim P(z_1 | u)$ and $2^{nR''_1}$ codewords $X^n_1 \sim P(x_1 | z_1, u)$.

Encoder 2 finds the associate bin from $Z^n_1$ and cooperatively transmits $x^n_2(m_{2,b}, u^n)$.

$X^n_1$ is determined by $(m'_1, b, m''_1, b)$ conditioned on $u^n$.

Decoding: assume $l_b$ is known, find $l_{b-1}, m_{1,b}, m_{2,b}$.
Achievability proof: Cooperative Binning [Kolte/Özgür/P.15]

- Bin all the typical set $\mathcal{T}_c^n(Z_1)$ into $2^{nH(Z_1|U)+\delta}$.
- For each bin $l$ generate codeword $u^n$ i.i.d. $\sim P(u)$.
- Split $R_1$ into $R_1 = R'_1 + R''_1$.
- Divide block of length $Bn$ into $B$ blocks of length $n$.
- For each $u^n$, generate $2^{nR'_1}$ codewords $z^n_1$ according to $\sim P(z_1|u)$ and $2^{nR''_1}$ codewords $X^n_1 \sim P(x_1|z_1, u)$.
- Encoder 2 finds the associate bin from $Z^n_1$ and cooperatively transmits $x^n_2(m_{2,b}, u^n)$.
- $X^n_1$ is determined by $(m'_{1,b}, m''_{1,b})$ conditioned on $u^n$.
- Decoding: assume $l_b$ is known, find $l_{b-1}, m_{1,b}, m_{2,b}$.
- $\forall l_{b-1}$ find unique $m'_{1,b}$ s.t. $\text{Bin}(z^n_b(m_{1,b'}|l_{b-1})) = l_b$. 

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Achievability proof: Cooperative Binning

- Bin all the typical set $\mathcal{T}_\epsilon^n(Z_1)$ into $2^{nH(Z_1|U)+\delta}$.
- For each bin $l$ generate codeword $u^n$ i.i.d. $\sim P(u)$.
- Split $R_1$ into $R_1 = R'_1 + R''_1$.
- Divide block of length $Bn$ into $B$ blocks of length $n$.
- For each $u^n$, generate $2^{nR'_1}$ codewords $z_1^n$ according to $\sim P(z_1|u)$ and $2^{nR''_1}$ codewords $X_1^n \sim P(x_1|z_1, u)$.
- Encoder 2 finds the associate bin from $Z_1^n$ and cooperatively transmits $x_2^n(m_{2, b}, u^n)$.
- $X_1^n$ is determined by $(m'_{1, b}, m''_{1, b})$ conditioned on $u^n$.
- Decoding: assume $l_b$ is known, find $l_{b-1}, m_{1, b}, m_{2, b}$.
- $\forall l_{b-1}$ find unique $m'_{1, b}$ s.t. $\text{Bin}(z^n_b(m_{1, b'}|l_{b-1})) = l_b$.
- The probability of finding the wrong $m'_{1, b}$:
  - There is more than one $z_1^n$ in bin $l_b$ for a given $l_{b-1}$.
Achievability proof: Cooperative Binning

- Bin all the typical set $\mathcal{T}_\epsilon^{(n)}(Z_1)$ into $2^{nH(Z_1|U)+\delta}$.
- For each bin $l$ generate codeword $u^n$ i.i.d. $\sim P(u)$.
- Split $R_1$ into $R_1 = R_1' + R_1''$.
- Divide block of length $Bn$ into $B$ blocks of length $n$.
- For each $u^n$, generate $2^{nR_1'}$ codewords $z^n_1$ according to $\sim P(z_1|u)$ and $2^{nR_1''}$ codewords $X^n_1 \sim P(x_1|z_1, u)$.
- Encoder 2 finds the associate bin from $Z_1^n$ and cooperatively transmits $x^n_2(m_{2,b}, u^n)$.
- $X^n_1$ is determined by $(m'_{1,b}, m''_{1,b})$ conditioned on $u^n$.
- Decoding: assume $l_b$ is known, find $l_{b-1}, m_{1,b}, m_{2,b}$.
- $\forall l_{b-1}$ find unique $m'_{1,b}$ s.t. $\text{Bin}(z^n_b(m_{1,b'}|l_{b-1})) = l_b$.
- The probability of finding the wrong $m'_{1,b}$:
  - There is more than one $z^n_1$ in bin $l_b$ for a given $l_{b-1}$. Number of bins $> 2^{nH(Z_1|U)}$. 

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Bin all the typical set $\mathcal{T}_\epsilon(n)(Z_1)$ into $2^{nH(Z_1|U) + \delta}$.

For each bin $l$ generate codeword $u^n$ i.i.d. $\sim P(u)$.

Split $R_1$ into $R_1 = R'_1 + R''_1$.

Divide block of length $Bn$ into $B$ blocks of length $n$.

For each $u^n$, generate $2^{nR'_1}$ codewords $z^n_1$ according to $\sim P(z_1|u)$ and $2^{nR''_1}$ codewords $X^n_1 \sim P(x_1|z_1,u)$.

Encoder 2 finds the associate bin from $Z^n_1$ and cooperatively transmits $x^n_2(m_{2,b}, u^n)$.

$X^n_1$ is determined by $(m'_1,b, m''_1,b)$ conditioned on $u^n$.

Decoding: assume $l_b$ is known, find $l_{b-1}, m_{1,b}, m_{2,b}$.

$\forall l_{b-1}$ find unique $m'_{1,b}$ s.t. Bin($z^n_b(m_{1,b'}|l_{b-1})$) = $l_b$.

The probability of finding the wrong $m'_{1,b}$:

- There is more than one $z^n_1$ in bin $l_b$ for a given $l_{b-1}$. Number of bins $> 2^{nH(Z_1|U)}$.
- $\hat{m}'_{1,b} \neq m'_{1,b}$ and $z^n_b(\hat{m}_{1,b'}|l_{b-1}) = z^n_b(m_{1,b'}|l_{b-1})$. 

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Achievability proof: Cooperative Binning [Kolte/Özgür/P.15]

- Bin all the typical set $\mathcal{T}_\epsilon^{(n)}(Z_1)$ into $2^{nH(Z_1|U)+\delta}$.
- For each bin $l$ generate codeword $u^n$ i.i.d. $\sim P(u)$.
- Split $R_1$ into $R_1 = R'_1 + R''_1$.
- Divide block of length $Bn$ into $B$ blocks of length $n$.
- For each $u^n$, generate $2^{nR'_1}$ codewords $z^n_1$ according to $\sim P(z_1|u)$ and $2^{nR''_1}$ codewords $X^n_1 \sim P(x_1|z_1, u)$.
- Encoder 2 finds the associate bin from $Z^n_1$ and cooperatively transmits $x^n_2(m_{2,b}, u^n)$.
- $X^n_1$ is determined by $(m'_{1,b}, m''_{1,b})$ conditioned on $u^n$.
- Decoding: assume $l_b$ is known, find $l_{b-1}, m_{1,b}, m_{2,b}$.
- $\forall l_{b-1}$ find unique $m'_{1,b}$ s.t. $\text{Bin}(z^n_b(m_{1,b'}|l_{b-1})) = l_b$.
- The probability of finding the wrong $m'_{1,b}$:
  - There is more than one $z^n_1$ in bin $l_b$ for a given $l_{b-1}$. Number of bins $> 2^{nH(Z_1|U)}$.
  - $\hat{m}'_{1,b} \neq m'_{1,b}$ and $z^n_b(\hat{m}_{1,b'}|l_{b-1}) = z^n_b(m_{1,b'}|l_{b-1})$.
  - $R'_1 \leq H(Z_1|U)$. 

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Bin all the typical set $\mathcal{T}_\epsilon(n)(Z_1)$ into $2^{nH(Z_1|U,S_1)} + \delta$.

For each bin $l$ generate codeword $u^n$ i.i.d. $\sim P(u)$.

Split $R_1$ into $R_1 = R'_1 + R''_1$.

Divide block of length $Bn$ into $B$ blocks of length $n$.

For each $u^n$, generate $2^{nR'_1}$ codewords $z^n_1$ according to $\sim P(z_1|u)$ and $2^{nR''_1}$ codewords $X^n_1 \sim P(x_1|z_1,u)$.

Encoder 2 finds the associate bin from $Z^n_1$ and cooperatively transmits $x^n_2(m_{2,b},u^n)$.

$X^n_1$ is determined by $(m'_{1,b}, m''_{1,b})$ conditioned on $u^n$.

Decoding: assume $l_b$ is known, find $l_{b-1}, m_{1,b}, m_{2,b}$.

$\forall l_{b-1}$ find unique $m'_{1,b}$ s.t. $\text{Bin}(z^n_b(m_{1,b'}|l_{b-1})) = l_b$.

The probability of finding the wrong $m'_{1,b}$:

- There is more than one $z^n_1$ in bin $l_b$ for a given $l_{b-1}$. Number of bins $> 2^{nH(Z_1|U)}$.

- $\hat{m}'_{1,b} \neq m'_{1,b}$ and $z^n_b(\hat{m}_{1,b'}|l_{b-1}) = z^n_b(m_{1,b'}|l_{b-1})$.

$R'_1 \leq H(Z_1|U)$. 

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Achievability proof: Cooperative Binning [Kolte/Özgür/P.15]

- Bin all the typical set $\mathcal{T}_\epsilon^{(n)}(Z_1)$ into $2^{nH(Z_1|U,S_1)} + \delta$.
- For each bin $l$ generate codeword $u^n$ i.i.d. $\sim P(u)$.
- Split $R_1$ into $R_1 = R'_1 + R''_1$.
- Divide block of length $Bn$ into $B$ blocks of length $n$.
- For each $u^n$, generate $2^{nR'_1}$ codewords $z^n_1$ according to $\sim P(z_1|u,s_1)$ and $2^{nR''}$ codewords $X^n_1 \sim P(x_1|z_1,u,s_1)$
- Encoder 2 finds the associate bin from $Z^n_1$ and cooperatively transmits $x^n_2(m_{2,b},u^n)$.
- $X^n_1$ is determined by $(m'_{1,b}, m''_{1,b})$ conditioned on $u^n$.
- Decoding: assume $l_b$ is known, find $l_{b-1}, m_{1,b}, m_{2,b}$.
- $\forall l_{b-1}$ find unique $m'_{1,b}$ s.t. $\text{Bin}(z^n_b(m_{1,b'}|l_{b-1})) = l_b$.
- The probability of finding the wrong $m'_{1,b}$:
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Achievability proof: Cooperative Binning [Kolte/Özgür/P.15]

- Bin all the typical set $\mathcal{T}_{\epsilon}^{(n)}(Z_1)$ into $2^{nH(Z_1|U,S_1)+\delta}$.
- For each bin $l$ generate codeword $u^n$ i.i.d. $\sim P(u)$.
- Split $R_1$ into $R_1 = R'_1 + R''_1$.
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- Decoding: assume $l_b$ is known, find $l_{b-1}, m_{1,b}, m_{2,b}$.
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A new coding scheme for cooperation
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  - $R'_1 \leq H(Z_1|U,S_1)$.
After error analysis we obtain

\[ R'_1 < H(Z_1 | U, S_1), \]
\[ R'_2 < H(Z_2 | U, S_2), \]
\[ R''_2 < I(X_2; Y | U, Z_2, X_1, S_1, S_2), \]
\[ R''_1 < I(X_1; Y | U, Z_1, X_2, S_1, S_2), \]
\[ R''_1 + R''_2 < I(X_1, X_2; Y | U, Z_1, Z_2, S_1, S_2), \]
\[ R_1 + R_2 < I(X_1, X_2; Y | S_1, S_2) \]
After error analysis we obtain

\[ R_1' < H(Z_1|U, S_1), \]
\[ R_2' < H(Z_2|U, S_2)), \]
\[ R_2'' < I(X_2; Y|U, Z_2, X_1, S_1, S_2), \]
\[ R_1'' < I(X_1; Y|U, Z_1, X_2, S_1, S_2)), \]
\[ R_1'' + R_2'' < I(X_1, X_2; Y|U, Z_1, Z_2, S_1, S_2), \]
\[ R_1 + R_2 < I(X_1, X_2; Y|S_1, S_2) \]

Using Fourier–Motzkin elimination we obtain:

**Theorem**

\[ R_1 \leq I(X_1; Y|U, X_2, Z_1, S_1, S_2) + H(Z_1|U, S_1), \]
\[ R_2 \leq I(X_2; Y|U, X_1, Z_2, S_1, S_2) + H(Z_2|U, S_2), \]
\[ R_1 + R_2 \leq I(X_1, X_2; Y|U, Z_1, Z_2, S_1, S_2) + H(Z_1, Z_2|U, S_1, S_2), \]
\[ R_1 + R_2 \leq I(X_1, X_2; Y|S_1, S_2), \]

for \( P(u)P(x_1|u, s_1)P(x_2|u, s_2) \) \[\text{[Kolte/Özgür/P.15]}\]
Fourier–Motzkin Elimination (FME)

- Eliminate unnecessary variables from linear inequalities
Fourier–Motzkin Elimination (FME)

- Eliminate unnecessary variables from linear inequalities
- Example

\[
R \leq R'' + H(Z|U) \\
R'' \leq I(X_2; Y|U, Z_2, X_1, S_1, S_2)
\]
Fourier–Motzkin Elimination (FME)

- Eliminate unnecessary variables from linear inequalities
- Example

\[
\begin{align*}
R'' & \geq R - H(Z|U) \\
R'' & \leq I(X_2; Y|U, Z_2, X_1, S_1, S_2)
\end{align*}
\]
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\[
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- FME: each lower bound less than each upper bound.

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- FME: each lower bound less than each upper bound.

\[
R - H(Z|U) \leq I(X_2; Y|U, Z_2, X_1, S_1, S_2)
\]

- Very useful in multi-user problems
- A computer can do it, but inserts many redundant inequalities.

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Open-source Matlab software www.ee.bgu.ac.il/~fmeit
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Applies identification of a redundant constraint:

$$\rho^* = \min_{x: A^{(i)}x \geq b^{(i)}} a_i^\top x$$

If $\rho^* \geq b_i$ then $a_i^\top x \geq b_i$ is a redundant constraint.
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Using Shannon-type inequalities:

$$\rho^* = \min_{\mathbf{h}: \mathbf{G}\mathbf{h} \geq 0} \mathbf{f}^\top \mathbf{h}$$

If $$\rho^* = 0$$ then $$\mathbf{f}^\top \mathbf{h} \geq 0$$.

- $$\mathbf{h}$$ - Vector with joint entropies (canonical form).
- $$\mathbf{G}\mathbf{h} \geq 0$$ - Elemental inequalities.
- $$\mathbf{Q}\mathbf{h} = 0$$ - Constraints due to PMF (e.g. Markov chains).
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Applies identification of a redundant constraint:

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- **h** - Vector with joint entropies (canonical form).
- **Gh ≥ 0** - Elemental inequalities.
- **Qh = 0** - Constraints due to PMF (e.g. Markov chains).

FME-IT combines the two LPs in one problem.
A new coding scheme for cooperation
MAC with Combined Cooperation & Partial Cribbing

[Kopetz/P./Shamai14]

Encoder 1

\[ Z_i = g_1(X_{1,i}) \]

Encoder 2

\[ X_{2,i}(m_2, m_{12}, Z_{i-1}) \]

Decoder

\[ Y_i \]

\[ (\hat{m}_1, \hat{m}_2) \]

\[ R_1 \leq I(X_1; Y|X_2, Z, U) + H(Z|U) + C_{12} \]

\[ R_2 \leq I(X_2; Y|X_1, U) \]

\[ R_1 + R_2 \leq I(X_1, X_2; Y|U, Z) + H(Z|U) + C_{12} \]

\[ R_1 + R_2 \leq I(X_1, X_2; Y) \]

for

\[ P(u)P(x_1, z|u)P(x_2|u)P(y|x_1, x_2) \]
Cooperative MAC with Oblivious Encoders [Kopetz/P./Shamai15]

Oblivious nodes: [Sanderovich/Shamai/Steinberg/Kramer08]
- Random codes
- Independent of the message and of each other.
Oblivious vs. Codebook-aware Encoding

Oblivious Encoding:

\[ R_1 \leq I(X_1; Y|X_2, Z, Q) + H(Z|Q) + C_{12}, \]
\[ R_2 \leq I(X_2; Y|X_1, Q), \]
\[ R_1 + R_2 \leq I(X_1, X_2; Y|Z, Q) + H(Z|Q), \]
\[ R_1 + R_2 \leq I(X_1, X_2; Y|Q), \]

for \( P(q)P(x_1|q)P(x_2|q)P(y|x_1, x_2). \)
Oblivious vs. Codebook-aware Encoding

Oblivious Encoding:

\[
\begin{align*}
R_1 &\leq I(X_1; Y|X_2, Z, Q) + H(Z|Q) + C_{12}, \\
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&\text{for } P(q)P(x_1|q)P(x_2|q)P(y|x_1, x_2).
\end{align*}
\]

Codebook-aware Encoding:

\[
\begin{align*}
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R_1 + R_2 &\leq I(X_1, X_2; Y|Q), \\
&\text{for } P(q)P(u|q)P(x_1|u, q)P(x_2|u, q)P(y|x_1, x_2).
\end{align*}
\]
Oblivious vs. Codebook-aware Encoding

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for \( P(q)P(u|q)P(x_1|u, q)P(x_2|u, q)P(y|x_1, x_2). \)

Bin-and-Forward vs Cooperative Binning
Additive Gaussian MAC with quantized cribbing

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Additive Gaussian MAC, No cribbing, $C_{12} = 0.3$

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A new coding scheme for cooperation
Two achievabilities scheme for MAC with partial (deterministic) cribbing:

1. Partial decoding
2. Cooperative binning

Cooperative binning solves asymmetric state at the TXs.
Cooperative binning solves semi-deterministic relay with state known to the source encoder and destination RX.

For oblivious encoder: Bin-and-forward

Many open problems: non causal partial cribbing, noisy cribbing, causal state at encoder only, ....
Summary

- Two achievabilities scheme for MAC with partial (deterministic) cribbing:
  1. Partial decoding
  2. Cooperative binning

- Cooperative binning solves asymmetric state at the TXs.
- Cooperative binning solves semi-deterministic relay with state known to the source encoder and destination RX.
- For oblivious encoder: Bin-and-forward
- Many open problems: non causal partial cribbing, noisy cribbing, causal state at encoder only, ....

*Thank you very much!*