2nd Semester 2010

Homework Set #5
Differential Entropy and Gaussian Channel

1. **Differential entropy.**
   Evaluate the differential entropy $h(X) = -\int f \ln f$ for the following:

   (a) Find the entropy of the exponential density $\lambda e^{-\lambda x}$, $x \geq 0$.

   (b) The sum of X_1 and X_2, where X_1 and X_2 are independent normal random variables with means μ_i and variances σ_i^2, $i = 1, 2$.

2. **Mutual information for correlated normals.** Find the mutual information $I(X;Y)$, where

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2 \left(0, \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix} \right).$$

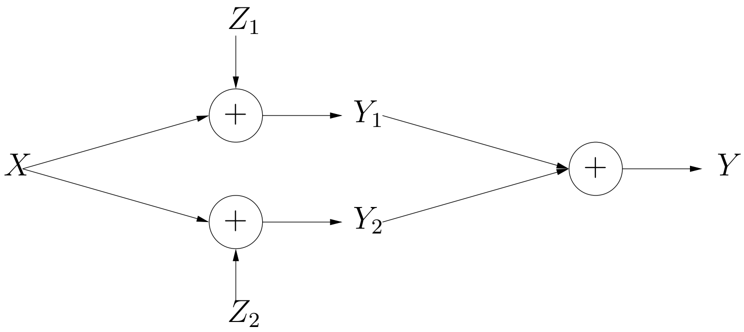
   Evaluate $I(X;Y)$ for $\rho = 1, \rho = 0$, and $\rho = -1$, and comment.

3. **Markov Gaussian mutual information.**
   Suppose that (X, Y, Z) are jointly Gaussian and that $X \to Y \to Z$ forms a Markov chain. Let X and Y have correlation coefficient ρ_1 and let Y and Z have correlation coefficient ρ_2. Find $I(X;Z)$.

4. **Output power constraint.**
   Consider an additive white Gaussian noise channel with an expected output power constraint P. (We might want to protect the eardrums of the listener.) Thus $Y = X + Z$, $Z \sim N(0, \sigma^2)$, Z is independent of X, and $EY^2 \leq P$. Assume $\sigma^2 < P$. Find the channel capacity.

5. **Multipath Gaussian channel.**
   Consider a Gaussian noise channel of power constraint P, where the

signal takes two different paths and the received noisy signals are added together at the antenna.



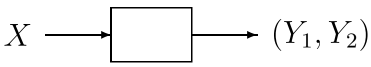
Let $Y = Y_1 + Y_2$ and $EX^2 \le P$.

(a) Find the capacity of this channel if Z_1 and Z_2 are jointly normal with covariance matrix

$$K = \begin{bmatrix} N & N\rho \\ N\rho & N \end{bmatrix}.$$

(b) What is the capacity for $\rho = 0, -1$, and 1 ?

6. **The two-look Gaussian channel.**



Consider the ordinary additive noise Gaussian channel with two correlated looks at X, i.e., $Y = (Y_1, Y_2)$, where

$$Y_1 = X + Z_1$$

$$Y_2 = X + Z_2$$

with a power constraint P on X, and $(Z_1, Z_2) \sim \mathcal{N}_2(\mathbf{0}, K)$, where

$$K = \begin{bmatrix} N & N\rho \\ N\rho & N \end{bmatrix}.$$

Find the capacity C for

(a) $\rho = 1$.

(b) $\rho = 0$.

(c) $\rho =$ -1.

Note that the capacity of the above channel in all cases is the same as the capacity of the channel $X \to Y_1 + Y_2$.

7. **Diversity System**

For the following system, a message $W \in \{1, 2, \ldots, 2^{nR}\}$ is encoded into two symbol blocks $X_1^n = (X_{1,1}, X_{1,2}, ..., X_{1,n})$ and $X_2^n = (X_{2,1}, X_{2,2}, ..., X_{2,n})$ that are being transmitted over a channel. The average power constrain on the inputs are $\frac{1}{n}E[\sum_{i=1}^{n} X_{1,i}^2] \leq P_1$ and $\frac{1}{n}E[\sum_{i=1}^{n} X_{2,i}^2] \leq P_2$. The channel has a multiplying effect on X_1, X_2 by factor h_1, h_2, respectively, i.e., $Y = h_1X_1 + h_2X_2 + Z$, where Z is a white Gaussian noise $Z \sim N(0, \sigma^2)$.

(a) Find the joint distribution of X_1 and X_2 that bring the mutual information $I(Y; X_1, X_2)$ to a maximum? (You need to find $\arg\max P_{X_1,X_2} I(X_1, X_2; Y)$.)

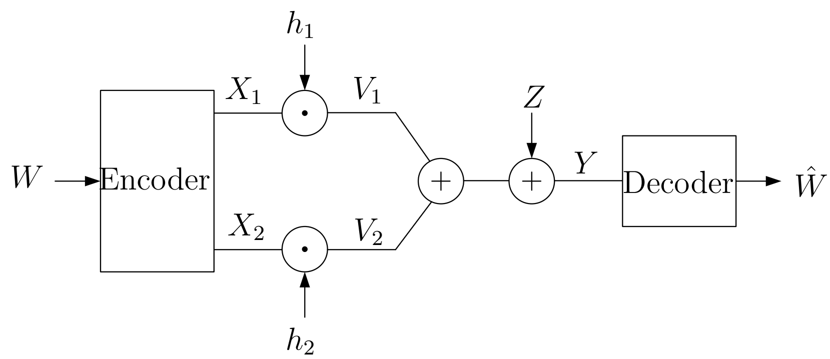


Figure 1: The communication model

(b) What is the capacity of the system ?

(c) Express the capacity for the following cases:

i. $h_1 = 1,\ h_2 = 1$?

ii. $h_1 = 1,\ h_2 = 0$?

iii. $h_1 = 0,\ h_2 = 0$?

8. **AWGN with two noises**

Figure 2 depicts a communication system with an AWGN (Additive white noise Gaussian) channel whith two i.i.d. noises $Z_1 \sim N(0, \sigma_1^2)$, $Z_2 \sim N(0, \sigma_2^2)$ that are independent of each other and are added to the signal X, i.e., $Y = X + Z_1 + Z_2$. The average power constrain on the input is P, i.e., $\frac{1}{n}E[\sum_{i=1}^n X_i^2] \leq P$. In the sub-questions below we consider the cases where the noise Z_2 may or may not be known to the encoder and decoder.

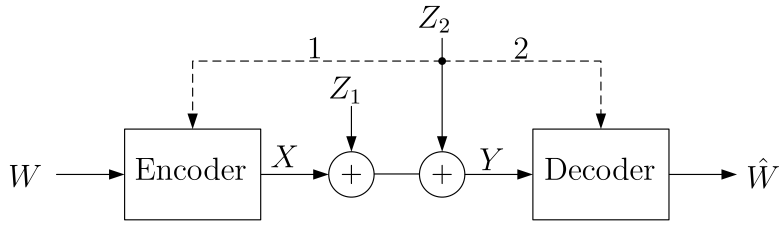


Figure 2: Two noise sources

(a) Find the channel capacity for the case in which the noise in not known to either sides (lines 1 and 2 are <u>disconnected</u> from the encoder and the decoder).

(b) Find the capacity for the case that the noise Z_2 is known to the encoder and decoder (lines 1 and 2 are <u>connected</u> to both the encoder and decoder). This means that the codeword X^n may depend on the message W and the noise Z_2^n and the decoder decision $\hat{W}$ may depend on the output Y^n and the noise Z_2^n. (**Hint**: Could the capacity be lager than $\frac{1}{2}\log(1 + \frac{P}{\sigma_1^2})$?)

(c) Find the capacity for the case that the noise Z_2 is known only to the decoder. (line 1 is <u>disconnected</u> from the encoder and line 2 is <u>connected</u> to the decoder). This means that the codewords X^n may depend only on the message W and the decoder decision $\hat{W}$ may depend on the output Y^n and the noise Z_2^n.

9. **Parallel channels and waterfilling**

Consider a pair of parallel Gaussian channels, i.e.,

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix},$$

where

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim \mathcal{N}\left(0, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}\right),$$

and there is a power constraint $E(X_1^2 + X_2^2) \leq P$. Assume that $\sigma_1^2 > \sigma_2^2$. At what power does the channel stop behaving like a single channel with noise variance σ_2^2, and begin behaving like a pair of channels, ie., at what power does the worst channel become useful?