1. Preprocessing the output.
   One is given a communication channel with transition probabilities
   \( p(y | x) \) and channel capacity \( C = \max_{p(x)} I(X; Y) \). A helpful statisti-
   cian preprocesses the output by forming \( \tilde{Y} = g(Y) \), yielding a channel
   \( p(\tilde{y} | x) \). He claims that this will strictly improve the capacity.
   
   (a) Show that he is wrong.
   (b) Under what conditions does he not strictly decrease the capacity?

2. The Z channel.
   The Z-channel has binary input and output alphabets and transition
   probabilities \( p(y | x) \) given by the following matrix:
   \[
   Q = \begin{bmatrix}
   1 & 0 \\
   1/2 & 1/2
   \end{bmatrix}
   \]
   \( x, y \in \{0, 1\} \)
   
   Find the capacity of the Z-channel and the maximizing input probabil-
   ity distribution.

3. Using two channels at once.
   Consider two discrete memoryless channels \((X_1, p(y_1 | x_1), Y_1)\) and
   \((X_2, p(y_2 | x_2), Y_2)\) with capacities \( C_1 \) and \( C_2 \) respectively. A new
   channel \((X_1 \times X_2, p(y_1 | x_1) \times p(y_2 | x_2), Y_1 \times Y_2)\) is formed in which
   \( x_1 \in X_1 \) and \( x_2 \in X_2 \), are simultaneously sent, resulting in \( y_1, y_2 \). Find
   the capacity of this channel.

4. A channel with two independent looks at Y.
   Let \( Y_1 \) and \( Y_2 \) be conditionally independent and conditionally identi-
   cally distributed given \( X \). Thus \( p(y_1, y_2 | x) = p(y_1 | x)p(y_2 | x) \).
   
   (a) Show \( I(X; Y_1, Y_2) = 2I(X; Y_1) - I(Y_1; Y_2) \).
   (b) Conclude that the capacity of the channel

   \[
   X \rightarrow (Y_1, Y_2)
   \]
is less than twice the capacity of the channel

\[ X \rightarrow Y_1 \]

5. **Choice of channels.**
Find the capacity \( C \) of the union of 2 channels \((X_1, p_1(y_1|x_1), Y_1)\) and \((X_2, p_2(y_2|x_2), Y_2)\) where, at each time, one can send a symbol over channel 1 or over channel 2 but not both. Assume the output alphabets are distinct and do not intersect.

(a) Show \( 2^C = 2^{C_1} + 2^{C_2} \).

(b) What is the capacity of this Channel?

\[ 1 \rightarrow 1 \]
\[ 2 \rightarrow 2 \]
\[ 3 \rightarrow 3 \]

6. **Cascaded BSCs.**
Consider the two discrete memoryless channels \((X, p_1(y|x), Y)\) and \((Y, p_2(z|y), Z)\).
Let \( p_1(y|x) \) and \( p_2(z|y) \) be binary symmetric channels with crossover probabilities \( \lambda_1 \) and \( \lambda_2 \) respectively.
(a) What is the capacity $C_1$ of $p_1(y|x)$?

(b) What is the capacity $C_2$ of $p_2(z|y)$?

(c) We now cascade these channels. Thus $p_3(z|x) = \sum_y p_1(y|x)p_2(z|y)$. What is the capacity $C_3$ of $p_3(z|x)$? Show $C_3 \leq \min\{C_1, C_2\}$.

(d) Now let us actively intervene between channels 1 and 2, rather than passively transmitting $y^n$. What is the capacity of channel 1 followed by channel 2 if you are allowed to decode the output $y^n$ of channel 1 and then reencode it as $\tilde{y}^n$ for transmission over channel 2? (Think $W \rightarrow x^n(W) \rightarrow y^n \rightarrow \tilde{y}^n(y^n) \rightarrow z^n \rightarrow \hat{W}$.)

(e) What is the capacity of the cascade in part c) if the receiver can view both $Y$ and $Z$?

7. Channel capacity

(a) What is the capacity of the following channel

```
               1
            /   \
           0     1
          / \   / \     \
         1   1\ 1/2/ 2
        /   /  \  \   \
       2   3 \ 1/2/ 4
       /     /   \
      3     4 \
```

(b) Provide a simple scheme that can transmit at rate $R = \log_2 3$ bits through this channel.