Homework Set #4
Channel and Source coding

1. **Preprocessing the output.**
One is given a communication channel with transition probabilities $p(y \mid x)$ and channel capacity $C = \max_{p(x)} I(X;Y)$. A helpful statistician preprocesses the output by forming $\tilde{Y} = g(Y)$, yielding a channel $p(\tilde{y}|x)$. He claims that this will strictly improve the capacity.

   (a) Show that he is wrong.

   (b) Under what conditions does he not strictly decrease the capacity?

2. **The Z channel.**
The Z-channel has binary input and output alphabets and transition probabilities $p(y|x)$ given by the following matrix:

$$Q = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} \qquad x, y \in \{0, 1\}$$

   Find the capacity of the Z-channel and the maximizing input probability distribution.

3. **Using two channels at once.**
Consider two discrete memoryless channels $(\mathcal{X}_1, p(y_1 \mid x_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p(y_2 \mid x_2), \mathcal{Y}_2)$ with capacities C_1 and C_2 respectively. A new channel $(\mathcal{X}_1 \times \mathcal{X}_2, p(y_1 \mid x_1) \times p(y_2 \mid x_2), \mathcal{Y}_1 \times \mathcal{Y}_2)$ is formed in which $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$, are *simultaneously* sent, resulting in y_1, y_2. Find the capacity of this channel.

4. **A channel with two independent looks at Y.**
Let Y_1 and Y_2 be conditionally independent and conditionally identically distributed given X. Thus $p(y_1, y_2|x) = p(y_1|x)p(y_2|x)$.

   (a) Show $I(X; Y_1, Y_2) = 2I(X; Y_1) - I(Y_1; Y_2)$.

   (b) Conclude that the capacity of the channel

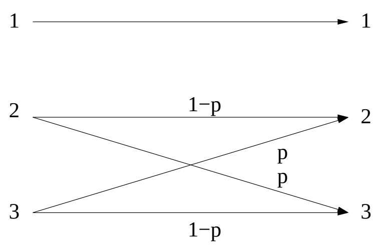
$$X \longrightarrow \boxed{} \longrightarrow (Y_1, Y_2)$$

is less than twice the capacity of the channel

5. **Choice of channels.**
Find the capacity C of the union of 2 channels $(\mathcal{X}_1, p_1(y_1|x_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p_2(y_2|x_2), \mathcal{Y}_2)$ where, at each time, one can send a symbol over channel 1 or over channel 2 but not both. Assume the output alphabets are distinct and do not intersect.

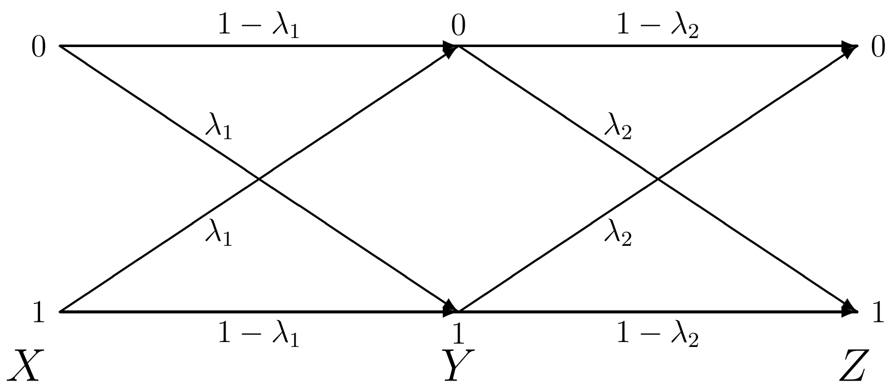
    (a) Show $2^C = 2^{C_1} + 2^{C_2}$.

    (b) What is the capacity of this Channel?



6. **Cascaded BSCs.**
Consider the two discrete memoryless channels $(\mathcal{X}, p_1(y|x), \mathcal{Y})$ and $(\mathcal{Y}, p_2(z|y), \mathcal{Z})$.

    Let $p_1(y|x)$ and $p_2(z|y)$ be binary symmetric channels with crossover probabilities λ_1 and λ_2 respectively.



(a) What is the capacity C_1 of $p_1(y|x)$?

(b) What is the capacity C_2 of $p_2(z|y)$?

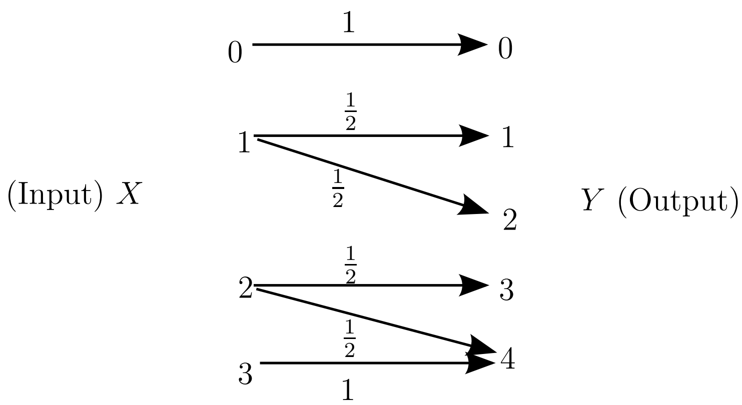
(c) We now cascade these channels. Thus $p_3(z|x) = \sum_y p_1(y|x)p_2(z|y)$. What is the capacity C_3 of $p_3(z|x)$? Show $C_3 \leq \min\{C_1, C_2\}$.

(d) Now let us actively intervene between channels 1 and 2, rather than passively transmitting y^n. What is the capacity of channel 1 followed by channel 2 if you are allowed to decode the output y^n of channel 1 and then reencode it as $\tilde{y}^n$ for transmission over channel 2? (Think $W \longrightarrow x^n(W) \longrightarrow y^n \longrightarrow \tilde{y}^n(y^n) \longrightarrow z^n \longrightarrow \hat{W}$.)

(e) What is the capacity of the cascade in part c) if the receiver can view *both* Y and Z?

7. **Channel capacity**

(a) What is the capacity of the following channel



(b) Provide a simple scheme that can transmit at rate $R = \log_2 3$ bits through this channel.