Multi-use Information Theory 2 December 27th, 2010

Lecture 11

Lecturer: Haim Permuter *Scribe: Iddo Naiss*

I. Relay with partial decode& forward

Here, we give an achievability proof for the following region.

Theorem 1 For a relay channel, the probability of error goes to zero if

$$R \leq \max_{p(x,x_1,u)} \{\min I(X,X_1;Y),\ I(U;Y_1|X_1) + I(X;Y|X_1,U)\}. \tag{1}$$

Proof: In a similar way to the proof given in previous class, we again divide the message of length N to B blocks of length n, and take B and n to infinity. Now, we divide each message M_b to two messages, $(M'_b,\ M''_b)$. In that case, $M'_b \in \{1,2,...,2^{nR'}\}$ and $M''_b \in \{1,2,...,2^{nR''}\}$ s.t. $R' + R'' = R$.

A. Code design

- Relay: Generate $2^{nR'}$ codewords- $x_1^n(m'_{b-1})$, where $x_1^n \sim p(x_1)$ i.i.d.
- Encoder:
  - For every $m'_{b-1} \in \{1,2,...,2^{nR'}\}$, generate $2^{nR'}$ codewords- $u^n(m'_{b-1}, m'_b)$, where $u^n \sim p(u|x_1)$ i.i.d.
  - Then, for every $(m'_b, m'_{b-1}) \in \{1,2,...,2^{nR'}\} \times \{1,2,...,2^{nR'}\}$, generate $2^{nR''}$ codewords- $x^n(m'_{b-1}, m'_b, m''_b)$, where $x^n \sim p(x|u,x_1)$ i.i.d.

B. Coding scheme

<u>In block b:</u>

- <u>Encoders</u>.
  - <u>Relay:</u> Given m'_{b-1} from previous block, the output of the relay is $x_1^n(m'_{b-1})$
  - <u>Transmitter:</u> First, given m'_{b-1}, m'_b, consider $u^n(m'_{b-1}, m'_b)$. Then, given m''_b as well, the output of the encoder is $x^n(u^n(m'_{b-1}, m'_b), m''_b)$.
- <u>Decoder</u>.
  - <u>Relay:</u> At the end of block b, given $\hat{m}'_{b-1}$, looks for $\hat{m}'_b$ s.t.

$$\left(x_1^n(\hat{m}'_{b-1}), u^n(\hat{m}'_{b-1}, \hat{m}'_b), y_1^n\right) \in T_\epsilon^n(X_1, U, Y_1). \tag{2}$$

- Receiver: At the end of block b, given $\hat{m}'_b$, looks for $\hat{m}''_b$ and $\hat{m}'_{b-1}$ s.t.

$$\big(x^n(\hat{m}'_{b-1}, \hat{m}'_b, \hat{m}''_b),\ x_1^n(\hat{m}'_{b-1}),\ u^n(\hat{m}'_{b-1}, \hat{m}'_b),\ y^n\big) \in T_\epsilon^n(X, X_1, U, Y). \tag{3}$$

C. Error analysis:

Error sets: Without loss of generality, assume that the message sent was $(m''_b,\ m'_b,\ m'_{b-1}) = (1, 1, 1)$

- $E_{Re,0}$: The sequence $u^n(\hat{m}'_{b-1} = 1, \hat{m}'_b = 1)$ does not satisfy (2).
- $E_{Re,1}$: There exists $j \neq 1$ for which the sequence $u^n(\hat{m}'_{b-1} = j, \hat{m}'_b = 1)$ satisfy (2).
- $E_{De,0}$: The sequences $\big(x^n(\hat{m}'_{b-1} = 1, \hat{m}'_b = 1, \hat{m}''_b = 1), u^n(\hat{m}'_{b-1} = 1, \hat{m}'_b = 1)\big)$ do not satisfy(3).
- E_{De}: There exists $(k, j) \neq (1, 1)$ for which the sequences $\big(x^n(\hat{m}'_{b-1} = j, \hat{m}'_b = 1, \hat{m}''_b = k), u^n(\hat{m}'_{b-1} = j, \hat{m}'_b)\big)$ satisfy (3). This part is divided to three cases.
  1) $E_{De,1}$: The output of the decoder is: $\hat{m}''_b \neq 1,\ \hat{m}'_{b-1} = 1$.
  2) $E_{De,1}$: The output of the decoder is: $\hat{m}''_b = 1,\ \hat{m}'_{b-1} \neq 1$.
  3) $E_{De,1}$: The output of the decoder is: $\hat{m}''_b \neq 1,\ \hat{m}'_{b-1} \neq 1$.

Probability of error: In the probability analysis we represent the message m'_{b-1} by a symbol j, and m''_b by k.

- $E_{Re,0}$: The probability of $E_{Re,0}$ goes to zero due to the law of large numbers.
- $E_{Re,1}$: In that case, the relay did not decode m'_b well.

$$\begin{aligned} p(E_{Re,1}) &= p\{\exists j \in \{2, 3, ..., 2^{nR'}\}\ s.t.\ (x_1^n(1), u^n(1, j), y_1^n) \in T_\epsilon^n(X_1, U, Y_1)\} \\ &\leq \sum_{j=2}^{2^{nR'}} p\{(x_1^n(1), u^n(1, j), y_1^n) \in T_\epsilon^n(X_1, U, Y_1)\} \\ &\leq \sum_{j=2}^{2^{nR'}} 2^{-n(I(U;Y_1|X_1)-\epsilon)} \\ &\leq 2^{n(R'-I(U;Y_1|X_1)+\epsilon)}. \end{aligned}$$

This probability goes to zero if

$$R' \leq I(U; Y_1|X_1). \tag{4}$$

- $E_{De,0}$: The probability of $E_{De,0}$ goes to zero due to the law of large numbers.
- E_{De}: We fully analyze $E_{De,i}$ for every $i = 1, 2, 3$.
  1) $E_{De,1}$: In that case, the decoder did not decode m''_b well.

$$p(E_{De,1}) = p\{\exists k \in \{2, 3, ..., 2^{nR''}\}\ s.t.\ (x^n(1, 1, k),\ x_1^n(1),\ u^n(1, 1),\ y^n) \in T_\epsilon^n(X, X_1, U, Y)\}$$

$$\leq \sum_{k=2}^{2^{nR''}} p\{(x^n(1,1,k),\ x_1^n(1),\ u^n(1,1),\ y^n) \in T(X,X_1,U,Y)\}$$

$$\leq \sum_{k=2}^{2^{nR''}} 2^{-n(I(X;Y|U,X_1)-\epsilon)}$$

$$\leq 2^{n(R''-I(X;Y|U,X_1)+\epsilon)}.$$

Thus, the probability goes to zero if

$$R'' \leq I(X;Y|U,X_1). \tag{5}$$

2) $E_{De,2}$: In that case, the decoder did not decode m'_{b-1} well.

$$p(E_{De,2}) = p\{\exists j \in \{2,3,...,2^{nR'}\}\ s.t.\ (x^n(j,1,1),\ x_1^n(j),\ u^n(j,1),\ y^n) \in T_\epsilon^n(X,X_1,U,Y)\}$$

$$\leq \sum_{j=2}^{2^{nR'}} p\{(x^n(j,1,1),\ x_1^n(j),\ u^n(j,1),\ y^n) \in T(X,X_1,U,Y)\}$$

$$\leq \sum_{j=2}^{2^{nR'}} 2^{-n(I(X,X_1,U;Y)-\epsilon)}$$

$$\leq 2^{n(R'-I(X,X_1,U;Y))+\epsilon)}.$$

Thus, the probability $p(E_{De,2})$ goes to zero if

$$R' \leq I(X,X_1,U;Y). \tag{6}$$

3) $E_{De,3}$: In that case, the decoder did not decode $m''_b,\ m'_{b-1}$ well.

$$p(E_{De,3}) = p\{\exists k,j \in \{2,3,...,2^{nR''}\} \times \{2,3,...,2^{nR'}\}\ s.t.$$
$$(x^n(j,1,k),\ x_1^n(j),\ u^n(j,1),\ y^n) \in T_\epsilon^n(X,X_1,U,Y)\}$$

$$\leq \sum_{k=2}^{2^{nR''}} \sum_{j=2}^{2^{nR'}} p\{(x^n(j,1,k),\ x_1^n(j),\ u^n(j,1),\ y^n) \in T(X,X_1,U,Y)\}$$

$$\leq \sum_{k=2}^{2^{nR''}} \sum_{j=2}^{2^{nR'}} 2^{-n(I(X,X_1,U;Y)-\epsilon)}$$

$$\leq 2^{n(R'+R''-I(X,X_1,U;Y))+\epsilon)}.$$

Thus, the probability $p(E_{De,3})$ goes to zero if

$$R' + R'' \leq I(X,X_1,U;Y). \tag{7}$$

First, Note that in order for this backwards decoding to work, we must always send $m'_B = 1$. This consideration does not effect the rate because the number of blocks B goes to infinity. Second, Note that $p(y|x, x_1, u) = p(y|x, x_1)$ due to the definition of the channel; hence $I(X, X_1, U; Y) = I(X, X_1; Y)$. Also note that condition (6) is included in condition (7). Now, we summarize. Using Fourier-Motzkin elimination, equations (4), (5), and $R = R' + R''$, will reduce to

$$R \leq I(U; Y_1|X_1) + I(X; Y|X_1, U). \tag{8}$$

Furthermore, equations (7) and $R = R' + R''$ will leave us with

$$R \leq I(X, X_1; Y). \tag{9}$$

Hence, we showed that if a rate R satisfies (1), the probability of error goes to zero, and the theorem is proven. ■

Let us mention a few remarks.

Remark 1 The division of each m_b to $(m'_b,\ m''_b)$ depends on the relay channel. The more the relay can send to the receiver, the bigger R' is.

Remark 2 In the previous lecture we gave an achievable rate for the relay channel in a different way, without dividing the message m_b. As a matter of fact, the coding scheme is the same, except that there we used $R'' = 0$, and hence $U = X$. If we apply the result here for $U = X$, then $I(X; Y|X_1, X) = 0$ and we have the following achievable rate:

$$R \leq \max_{p(x,x_1)} \{\min I(X, X_1; Y),\ I(X; Y_1|X_1)\}.$$

This bound is the same as given in the previous lecture. Note that this bound was shown by Cover and El Gamal [1, Theorem 1] to be the capacity of the relay channel, where the relay channel is physically degraded, i.e.,

$$p(y, y_1|x, x_1) = p(y_1|x, x_1)p(y|y_1, x_1).$$

They used a 'decode and forward' method, instead of the partial one, as we can see when $R'' = 0$.

Remark 3 Another simple case is when the relay can send only little information to the decoder. In that case we don't want to use it at all, i.e., $R' = 0$ and $U = X_1 = \emptyset$. Therefore, we are left with one constraint, which is

$$C \geq \max_{p(x)} I(X; Y).$$

Not surprisingly, this is what we would expect if we take the relay out of the model.

Remark 4 The 'partial decode and forward' method was used in [1, Theorem 7], where the model is more complex. If the relay generates some $\hat{Y}_1$, and an auxiliary r.v. V generates the input X_1 of the channel, then an achievable rate is

$$R_0 = \max\min\{I(X;Y,\hat{Y}_1|X_1,U) + I(U;Y_1|X_1;V),\ I(X,X_1;Y) - I(\hat{Y}_1;Y_1|X_1,X,U,Y)\}.$$

Setting $\hat{Y}_1 = \emptyset,\ V = \emptyset$, we are left with the expression in (1). This was mentioned by El Gamal and Aref [2] later on.

REFERENCES

[1] T. M. Cover and A El Gamal. Capacity theorems for the relay channel. *IEEE Trans. Inf. Theory,* IT-25:572-584,1979.

[2] A. El Gamal and M. Aref. The capacity of the semi-deterministic relay channel. *IEEE Trans. Inf. Theory,* vol. IT-28, no. 3, p. 536, May 1982.