Multi-use Information Theory 2 November 23th, 2010

# Lecture by Alon Orlitsky

*Lecturer: Haim Permuter* *Scribe: Iddo Naiss*

## I. A QUESTION BY PROF ALON ORLITSKY [1]

Let $n$ be the number of shirts in the store, and let $k < n$ be a constant number of shirts that each person likes. Note that there are exactly $\binom{n}{k}$ of such subsets. Say that a person wants to send a message to the store in order to receive one of the shirts he likes. What is the rate with which we needs to use in order to follow this rule (worst case)?

First, we wish to define this question as a rate distortion problem. For that purpose we define $[n] = \{1, 2, ..., n\}$, and

$$\mathcal{X} = [n]^k = \{s : s \subseteq [n],\ |s| = k\}. \tag{1}$$

This imply $|\mathcal{X}| = \binom{n}{k}$. Furthermore, we define $\hat{\mathcal{X}} = [n]$. The distortion function will be

$$d(x, \hat{x}) = \begin{cases} 0, & \hat{x} \in\ x \\ \infty, & \text{otherwise} \end{cases}$$

*Answer:* In order to gain intuition we first discuss upper bounds.

1) If we want to describe each subset be a stream of bits, we need to describe $\binom{n}{k}$ subsets, and thus the rate will be $\log \binom{n}{k} \approx k \log n - \log k!$.
2) We can reduce it further if we only ask for one of the shirts, thus we need to describe only $n$ items and the rate will be $\log n$.
3) An improvement will be to send the minimal number in the subset, hence describing only $n - k + 1$ items and the rate will be $\log n - k + 1$.

We study the last bound.

- For $k = 1$, i.e., a person likes only one shirt, then he must tell the store owner exactly one of $n$ shirts. In that case, the rate will be $\log n$ as expected.
- For $k = n$, i.e., a person likes all shirts, then he doesn't have to send any bits at all. In that case, the rate will be $\log 1 = 0$ as expected.

Further analysis will show that the last bound is indeed tight.

We can expand the question in the following way. Assume that now the store has $n$ pants, and the number of items a person likes is also $k$. What is the rate now in order to get a shirt and a pants the a person likes? *Answer:* Again, we start by giving the upper bounds.

1) Clearly, $2\log n - k + 1$ works since we can send 2 streams, 1 for each type.
2) If we look at the case where $k > n/2$, then the subset of shirts and the subset of pants must have a joint number, and so we can send it. In that case we have to describe $n$ items, hence the rate is $\log n$.

Is that the best we can do? What is the answer when $k \leq n/2$? What is the solution If the store possesses $l$ types of items? ! Good Luck!

## II. Exercise

The analysis given by Prof. Alon Orlitsky is the worst case. However, if

$$d(x, \hat{x}) = \begin{cases} 0, & \hat{x} \in \ x \\ 1, & \text{otherwise} \end{cases}$$

what is $R(D = 0)$?

## References

[1] A. Orlitsky, Scalar vs. vector quantization: worst-case analysis, IEEE Transactions on Information Theory, 48:6 (June 2001), pp. 1393-1409.