Homework Set #3
Broadcast channel, Marton’s region, Semi-deterministic BC

1. Degraded Erasure Broadcast Channel. Consider the following degraded broadcast channel.

(a) What is the capacity of the channel from \( X \) to \( Y_1 \)?
(b) From \( X \) to \( Y_2 \)?
(c) What is the capacity region of all \((R_1, R_2)\) achievable for this broadcast channel? Simplify and sketch.

2. Deterministic broadcast channel.

A deterministic broadcast channel is defined by an input \( X \), two outputs, \( Y_1 \) and \( Y_2 \) which are functions of the input \( X \). Thus \( Y_1 = f_1(X) \) and \( Y_2 = f_2(X) \). Let \( R_1 \) and \( R_2 \) be the rates at which information can be sent to the two receivers.

- Prove that
  
  \[
  R_1 \leq H(Y_1) \quad \text{(1)}
  \]
  
  \[
  R_2 \leq H(Y_2) \quad \text{(2)}
  \]
  
  \[
  R_1 + R_2 \leq H(Y_1, Y_2) \quad \text{(3)}
  \]

- Suggest what would be the capacity region of the deterministic broadcast channel.
• Prove the achievability of the region you have suggested. (Hint: you may use Marton achievable region.)

A semi deterministic broadcast channel is defined by an input $X$, two outputs, $Y_1$ and $Y_2$ where $Y_1$ is function of the input $X$, i.e., $Y_1 = f_1(X)$, and $Y_2$ is determined by a memoryless channel $P_{Y_2|X}$. Let $R_1$ and $R_2$ be the rates at which information can be sent to the two receivers.

Prove that the capacity region is the set of $R_1, R_2$ that satisfies

$$R_1 \leq H(Y_1)$$
$$R_2 \leq I(U; Y_2)$$
$$R_1 + R_2 \leq I(U; Y_2) + H(Y_1|U)$$

4. Mutual Covering Lemma: Prove the following result.
Let $(U_1, U_2) \sim p(u_1, u_2)$ and $\epsilon > 0$. Let $U_1^n(m_1), m_1 \in [1, ..., 2^{nr_1}]$, be pairwise independent random sequences, each distributed according to $\prod_{i=1} P_{U_1}(u_{1,i})$. Similarly, Let $U_2^n(m_2), m_2 \in [1, ..., 2^{nr_2}]$, be pairwise independent random sequences, each distributed according to $\prod_{i=1} P_{U_2}(u_{2,i})$. Assume that $U_1^n(m_1) : m_1 \in [1, ..., 2^{nr_1}]$ and $U_2^n(m_2) : m_2 \in [1, ..., 2^{nr_2}]$ are independent.

Then, there exists $\delta(\epsilon)$ that goes to 0 as $\epsilon \to 0$ such that if

$$r_1 + r_2 > I(U_1; U_2) + \delta(\epsilon),$$

then

$$\lim_{n \to \infty} \Pr\{(U_1^n(m_1), U_2^n(m_2)) \notin T_\epsilon(U_1, U_2) \forall m_1 \in [1, ..., 2^{nr_1}], m_2 \in [1, ..., 2^{nr_2}]\} = 0$$

In addition to the prove, please explain, how it extends the covering lemma.