Homework Set #2
Broadcast channel, degraded message set, Csiszar Sum Equality

1. Convexity of capacity region of broadcast channel. Let $C \subseteq \mathbb{R}^2$ be the capacity region of all achievable rate pairs $R = (R_1, R_2)$ for the broadcast channel. Show that $C$ is a convex set by using a timesharing argument.

Specifically, show that if $R^{(1)}$ and $R^{(2)}$ are achievable, then $\lambda R^{(1)} + (1 - \lambda)R^{(2)}$ is achievable for $0 \leq \lambda \leq 1$.

2. Joint typicality Let $x^n, y^n$ be jointly strong-typical i.e., $(x^n, y^n) \in T_{e}^{(n)}(X,Y)$, and let $Z^n$ be distributed according to $\prod_{i=1}^{n} p_{Z|X}(z_i|x_i)$ (instead of $p_{Z|X,Y}(z_i|x_i, y_i)$). Then, $P\{(x^n, y^n, Z^n) \in T_{e}^{(n)}(X,Y,Z)\} \leq 2^{-n(I(Y;Z|X) - \delta(\epsilon))}$, where $\delta(\epsilon) \to 0$ when $\epsilon \to 0$.

3. Broadcast capacity depends only on the conditional marginals. Consider the general broadcast channel $(X, Y_1 \times Y_2, p(y_1, y_2 \mid x))$. Show that the capacity region depends only on $p(y_1 \mid x)$ and $p(y_2 \mid x)$. To do this, for any given $((2^{nR_1}, 2^{nR_2}), n)$ code, let

\begin{align*}
P_1^{(n)} &= P\{\hat{W}_1(Y_1) \neq W_1\}, \quad (1) \\
P_2^{(n)} &= P\{\hat{W}_2(Y_2) \neq W_2\}, \quad (2) \\
P^{(n)} &= P\{(\hat{W}_1, \hat{W}_2) \neq (W_1, W_2)\}. \quad (3)
\end{align*}

Then show

$$\max\{P_1^{(n)}, P_2^{(n)}\} \leq P^{(n)} \leq P_1^{(n)} + P_2^{(n)}.$$

The result now follows by a simple argument.

Remark: The probability of error $P^{(n)}$ does depend on the conditional joint distribution $p(y_1, y_2 \mid x)$. But whether or not $P^{(n)}$ can be driven to zero (at rates $(R_1, R_2)$) does not (except through the conditional marginals $p(y_1 \mid x), p(y_2 \mid x)$).

4. Degraded broadcast channel. Find the capacity region for the degraded broadcast channel in following figure.
5. **Csiszar Sum Equality.** Let $X^n$ and $Y^n$ be two random vectors with arbitrary joint probability distribution. Show that:

\[
\sum_{i=1}^{n} I(X_i^n; Y_i|Y_i^{i-1}) = \sum_{i=1}^{n} I(Y_i^{i-1}; X_i|X_{i+1}^n) \tag{4}
\]

As we shall see this inequality is useful in proving converses to several multiple user channels. (Hint: You can prove this by induction or by expanding the terms on both sides using the chain rule.)

6. **Broadcast Channel with Degraded Message Sets.** Consider a general DM broadcast channel $(X; p(y_1, y_2|x); Y_1; Y_2)$. The sender $X$ encodes two messages $(W_0; W_1)$ uniformly distributed over \{1, 2, ..., $2^{nR_0}$\} and \{1, 2, ..., $2^{nR_1}$\}. Message $W_0$ is to be sent to both receivers, while message $W_1$ is only intended for receiver $Y_1$.

The capacity region is given by the set $C$ of all $(R_0; R_1)$ such that:

\[
R_0 \leq I(U; Y_2) \tag{5}
\]
\[
R_1 \leq I(X; Y_1|U) \tag{6}
\]
\[
R_0 + R_1 \leq I(X; Y_1) \tag{7}
\]

for some $p(u)p(x|u)$.

(a) Show that the set $C$ is convex.
(b) Provide the achievability proof
(c) Provide a converse proof. You may derive your own converse or use the steps below.

• An alternative characterization of the capacity region is the set $C'$ of all $(R_0, R_1)$ such that:

\[
\begin{align*}
R_0 &\leq \min\{I(U; Y_1); I(U; Y_2)\} \\
R_0 + R_1 &\leq \min\{I(X; Y_1); I(X; Y_1|U) + I(U; Y_2)\}
\end{align*}
\]  

for some $p(u)p(x|u)$. Show that $C = C'$.

• Define $U_i = (W_0; Y_{i-1}^i, Y_{2,i+1}^n)$. Show that

\[
n(R_0 + R_1) \leq \sum_{i=1}^n (I(X_i; Y_{1,i}|U_i) + I(U_i; Y_{2,i})) + n\epsilon_n \tag{10}
\]

using the steps

\[
n(R_0 + R_1) \leq I(W_1; Y_1^n|W_0) + I(W_0; Y_2^n) + n\epsilon_n
\leq \sum_{i=1}^n I(X_i; Y_{1,i}|U_i) + I(Y_{2,i+1}^n; Y_{2,i}|W_0, Y_1^{i-1})
+ I(U_i; Y_{2,i}) - I(Y_{1}^{i-1}; Y_{2,i}|W_0, Y_{2,(i+1)}^n) + n\epsilon_n. \tag{11}
\]

Then use the identity from previous exercise to cancel the second and fourth terms.