1. **Rate distortion for uniform source with Hamming distortion**.

Consider a source $X$ uniformly distributed on the set $\{1, 2, \ldots, m\}$. Find the rate distortion function for this source with Hamming distortion, i.e.,

$$d(x, \hat{x}) = \begin{cases} 0 & \text{if } x = \hat{x}, \\ 1 & \text{if } x \neq \hat{x}. \end{cases}$$

2. **Erasure distortion**

Consider $X \sim \text{Bernoulli}(\frac{1}{2})$, and let the distortion measure be given by the matrix

$$d(x, \hat{x}) = \begin{bmatrix} 0 & 1 & \infty \\ \infty & 1 & 0 \end{bmatrix}. \quad (1)$$

Calculate the rate distortion function for this source. Can you suggest a simple scheme to achieve any value of the rate distortion function for this source?

3. **Rate distortion.**

A memoryless source $U$ is uniformly distributed on $\{0, \ldots, r-1\}$. The following distortion function is given by

$$d(u, v) = \begin{cases} 0, & u = v, \\ 1, & u = v \pm 1 \mod r, \\ \infty, & \text{otherwise}. \end{cases}$$

Show that the rate distortion function is

$$R(D) = \begin{cases} \log r - D - h_2(D), & D \leq \frac{2}{3}, \\ \log r - \log 3, & D > \frac{2}{3}. \end{cases}$$
4. **Adding a column to the distortion matrix.** Let $R(D)$ be the rate distortion function for an i.i.d. process with probability mass function $p(x)$ and distortion function $d(x, \hat{x})$, $x \in \mathcal{X}$, $\hat{x} \in \hat{\mathcal{X}}$. Now suppose that we add a new reproduction symbol $\hat{x}_0$ to $\hat{\mathcal{X}}$ with associated distortion $d(x, \hat{x}_0)$, $x \in \mathcal{X}$. Can this increase $R(D)$? Explain.

5. **Simplification.** Suppose $\mathcal{X} = \{1, 2, 3, 4\}$, $\hat{\mathcal{X}} = \{1, 2, 3, 4\}$, $p(i) = \frac{1}{4}$, $i = 1, 2, 3, 4$, and $X_1, X_2, \ldots$ are i.i.d. $\sim p(x)$. The distortion matrix $d(x, \hat{x})$ is given by

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(a) Find $R(0)$, the rate necessary to describe the process with zero distortion.

(b) Find the rate distortion function $R(D)$.

(Hint: The distortion measure allows to simplify the problem into one you have already seen.)

(c) Suppose we have a nonuniform distribution $p(i) = p_i$, $i = 1, 2, 3, 4$. What is $R(D)$?

6. **Rate distortion for two independent sources.** Can one simultaneously compress two independent sources better than compressing the sources individually? The following problem addresses this question. Let the pair $\{(X_i, Y_i)\}$ be iid $\sim p(x, y)$. The distortion measure for $X$ is $d(x, \hat{x})$ and its rate distortion function is $R_X(D)$. Similarly, the distortion measure for $Y$ is $d(y, \hat{y})$ and its rate distortion function is $R_Y(D)$.

Suppose we now wish to describe the process $\{(X_i, Y_i)\}$ subject to distortion constraints $\lim_{n \to \infty} Ed(X^n, \hat{X}^n) \leq D_1$ and $\lim_{n \to \infty} Ed(Y^n, \hat{Y}^n) \leq D_2$. Our rate distortion theorem can be shown to naturally extend to this setting and imply that the minimum rate required to achieve these distortion is given by

$$R_{X,Y}(D_1, D_2) = \min_{p(\hat{x}, \hat{y}|x,y): Ed(X,X) \leq D_1, Ed(Y,Y) \leq D_2} I(X, Y; \hat{X}, \hat{Y})$$
Now, suppose the \( \{X_i\} \) process and the \( \{Y_i\} \) process are independent of each other.

(a) Show
\[
R_{X,Y}(D_1, D_2) \geq R_X(D_1) + R_Y(D_2).
\]
(b) Does equality hold?

7. **One bit quantization of a single Gaussian random variable.**

Let \( X \sim \text{Norm}(0, \sigma^2) \) and let the distortion measure be squared error. Here we do not allow block descriptions. Show that the optimum reproduction points for 1 bit quantization are \( \pm \sqrt{\frac{2}{\pi}} \sigma \), and that the expected distortion for 1 bit quantization is \( \frac{2}{\pi} \sigma^2 \).

Compare this with the distortion rate bound \( D = \sigma^2 2^{-2R} \) for \( R = 1 \).

8. **Side information.**

A memoryless source generates i.i.d. pairs of random variables \((U_i, S_i), \ i = 1, 2, \ldots \) on finite alphabets, according to
\[
p(u^n, s^n) = \prod_{i=1}^{n} p(u_i, s_i).
\]

We are interested in describing \( U \), when \( S \), called side information, is known both at the encoder and at the decoder. Prove that the rate distortion function is given by
\[
R_{U|S}(D) = \min_{p(v|u,s):E[d(U,V)]\leq D} I(U;V|S).
\]

Compare \( R_{U|S}(D) \) with the ordinary rate distortion function \( R_U(D) \) without any side information. What can you say about the influence of the correlation between \( U \) and \( S \) on \( R_{U|S}(D) \)?

9. **Rate distortion to two users.** (50 points)

Consider a rate distortion problem where the source \( \{X_i\}_{i \geq 1} \) is i.i.d. distributed according to \( P(x) \) and a single message \( T(X^n) \in \{1, 2, \ldots, 2^{nR}\} \) is sent in blocks of length \( n \) to two users \( Y \) and \( Z \). The setting is illustrated in Fig. 1. The goal is that the reconstruction by the two users
will satisfy a distortion constraint

\[
E \left[ \frac{1}{n} \sum_{i=1}^{n} d_y(X_i, Y_i) \right] \leq D_y,
\]

\[
E \left[ \frac{1}{n} \sum_{i=1}^{n} d_z(X_i, Z_i) \right] \leq D_z,
\]

for some large \( n \), where \( Y_i \) and \( Z_i \) are the reconstructions at time \( i \) at User \( Y \) and User \( Z \), respectively.

(a) Find the minimum rate that can achieve the goal.

(b) Prove that such a code exists (achievability).

(c) Prove that does not exist a code with a lower rate than the one you found in a) that achieves the goal (converse).

(d) Consider the case where \( X \) is a Gaussian random variable with variance \( \sigma_x^2 \) and the distortions are the square error, i.e.,

\[
d_y(X_i, Y_i) = (X_i - Y_i)^2,
\]

\[
d_z(X_i, Z_i) = (X_i - Z_i)^2.
\]

\( (3) \)

i. If \( D_z = \infty \) and \( D_y < \infty \), what is the minimum achievable rate?

ii. If \( D_z < \infty \) and \( D_y < \infty \), what is the minimum achievable rate?
(e) What is the rate needed to achieve a specific coordination $P(x)P(y, z|x)$. Prove it.

(f) Now consider the case that there is additional rate $R'$ that is sent from the encoder to User $Z$ only. This is shown in Fig. 2. Let the achievable rate region be the set of all pair-rates that achieve the distortion constraint. Define the code mathematically, find the achievable rate region, and provide a complete prove.