Homework Set #2

1. **Csiszar Sum Equality**. Let X^n and Y^n be two random vectors with arbitrary joint probability distribution. Show that:

$$\sum_{i=1}^{n} I(X_{i+1}^n; Y_i | Y^{i-1}) = \sum_{i=1}^{n} I(Y^{i-1}; X_i | X_{i+1}^n) \quad (1)$$

   As we shall see this inequality is useful in proving converses to several multiple user channels. (Hint: You can prove this by induction or by expanding the terms on both sides using the chain rule.)

2. **Gelfand Pinsker.** Consider the point-to-point channel $P(y|x,s)$ with state information where the state $\{S_i\}_{i\geq 1}$ is i.i.d and is known to the encoder. Define the setting and prove that the capacity is

$$C = \max_{P(s)P(u|s)P(x|u,s)P(y|x,s)} I(U;Y) - I(U;S).$$

   Prove the achievability and the converse.**(10 points bonus if a student proves the converse differently from what exists in the lecture notes.)**

3. **Degraded BC with common message** Consider the physical degraded broadcast channel that we learned in class, with an additional common message at rate R_0 that should be decoded at three users. Namely, there are three rates (R_0, R_1, R_2). What is the capacity?

4. **BC with only common message** Consider a general BC with two users where only a common message is sent. Namely, there exists only one message at rate R_0 that needs to be decoded at Decoder 1 and 2. What is the capacity region.

5. **Multiple user BC** Extend the physical degraded broadcast channel to $K \geq 2$ users. Define the problem, and derive the capacity region.

6. **Broadcast capacity depends only on the conditional marginals.** Consider the general broadcast channel $(X, Y_1 \times Y_2, p(y_1, y_2 \mid x))$.

   (a) Show that the capacity region without feedback depends only on $p(y_1 \mid x)$ and $p(y_2 \mid x)$. To do this, for any given $((2^{nR_1}, 2^{nR_2}), n)$ code, let

$$P_1^{(n)} = P\{\hat{W}_1(\mathbf{Y}_1) \neq W_1\}, \tag{2}$$

$$P_2^{(n)} = P\{\hat{W}_2(\mathbf{Y}_2) \neq W_2\}, \tag{3}$$

$$P^{(n)} = P\{(\hat{W}_1, \hat{W}_2) \neq (W_1, W_2)\}. \tag{4}$$

   Then show

$$\max\{P_1^{(n)}, P_2^{(n)}\} \leq P^{(n)} \leq P_1^{(n)} + P_2^{(n)}.$$

   The result now follows by a simple argument.

   Remark: The probability of error $P^{(n)}$ *does* depend on the conditional joint distribution $p(y_1, y_2 \mid x)$. But whether or not $P^{(n)}$ can be driven to zero (at rates (R_1, R_2)) *does not* (except through the conditional marginals $p(y_1 \mid x), p(y_2 \mid x)$).

   (b) Is the claim above true also when there is feedback from the decoders to the encoder?

7. **Degraded broadcast channel.** Find the capacity region for the degraded broadcast channel in Fig. 1.

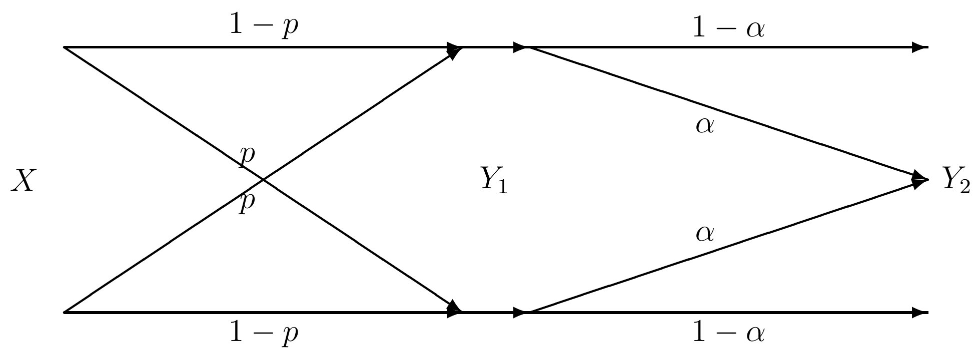


Figure 1: Broadcast channel with a binary symmetric channel and an erasure channel

8. **Broadcast Channel with Degraded Message Sets.** Consider a general DM broadcast channel $(\mathcal{X}; p(y_1, y_2|x); \mathcal{Y}_1; \mathcal{Y}_2)$. The sender X encodes two messages $(W_0; W_1)$ uniformly distributed over $\{1, 2..., 2^{nR_0}\}$ and $\{1, 2, ...2^{nR_1}\}$. Message W_0 is to be sent to both receivers, while message W_1 is only intended for receiver Y_1.

   The capacity region is given by the set $\mathcal{C}$ of all $(R_0; R_1)$ such that:

$$\begin{aligned} R_0 &\leq I(U; Y_2) & (5)\\ R_1 &\leq I(X; Y_1|U) & (6)\\ R_0 + R_1 &\leq I(X; Y_1) & (7) \end{aligned}$$

   for some $p(u)p(x|u)$.

   (a) Show that the set $\mathcal{C}$ is convex.

   (b) Provide the achievability proof

   (c) Provide a converse proof. You may derive your own converse or use the steps below.

   - An alternative characterization of the capacity region is the set $\mathcal{C}'$ of all (R_0, R_1) such that:

$$\begin{aligned} R_0 &\leq \min\{I(U; Y_1); I(U; Y_2)\} & (8)\\ R_0 + R_1 &\leq \min\{I(X; Y_1); I(X; Y_1|U) + I(U; Y_2)\} & (9) \end{aligned}$$

     for some $p(u)p(x|u)$. Show that $\mathcal{C} = \mathcal{C}'$.

   - Define $U_i = (W_0; Y_1^{i-1}, Y_{2,i+1}^n)$. Show that

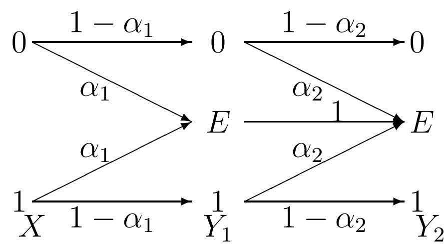
$$n(R_0 + R_1) \leq \sum_{i=1}^{n} (I(X_i; Y_{1,i}|U_i) + I(U_i; Y_{2,i})) + n\epsilon_n \qquad (10)$$

     using the steps

$$\begin{aligned} n(R_0 + R_1) &\leq I(W_1; Y_1^n|W_0) + I(W_0; Y_2^n) + n\epsilon_n\\ &\leq \sum_{i=1}^{n} I(X_i; Y_{1,i}|U_i) + I(Y_{2,i+1}^n; Y_{1,i}|W_0.Y_1^{i-1})\\ &\quad + I(U_i; Y_{2,i}) - I(Y_1^{i-1}; Y_{2,i}|W_0, Y_{2,(i+1)}^n) + n\epsilon_n. \end{aligned} \qquad (11)$$

Then use the identity from previous exercise to cancel the second and fourth terms.

9. **Degraded Erasure Broadcast Channel.** Consider the following degraded broadcast channel.



   (a) What is the capacity of the channel from X to Y_1?

   (b) From X to Y_2?

   (c) What is the capacity region of all (R_1, R_2) achievable for this broadcast channel? Simplify and sketch.

10. **Deterministic broadcast channel.**

    A deterministic broadcast channel is defined by an input X, two outputs, Y_1 and Y_2 which are functions of the input X. Thus $Y_1 = f_1(X)$ and $Y_2 = f_2(X)$. Let R_1 and R_2 be the rates at which information can be sent to the two receivers.

    - Prove that

$$R_1 \leq H(Y_1) \tag{12}$$
$$R_2 \leq H(Y_2) \tag{13}$$
$$R_1 + R_2 \leq H(Y_1, Y_2) \tag{14}$$

    - Suggest what would be the capacity region of the deterministic broadcast channel.
    - Prove the achievability of the region you have suggested. (Hint: you may use Marton achievable region.)

11. **Semi-Deterministic broadcast channel.**

A semi deterministic broadcast channel is defined by an input X, two outputs, Y_1 and Y_2 where Y_1 is function of the input X, i.e., $Y_1 = f_1(X)$, and Y_2 is determined by a memoryless channel $P_{Y_2|X}$. Let R_1 and R_2 be the rates at which information can be sent to the two receivers.

Prove that the capacity region is the set of R_1, R_2 that satisfies

$$R_1 \leq H(Y_1) \quad (15)$$

$$R_2 \leq I(U; Y_2) \quad (16)$$

$$R_1 + R_2 \leq I(U; Y_2) + H(Y_1|U) \quad (17)$$

12. **Mutual Covering Lemma**: Prove the following result.

Let $(U_1, U_2) \sim p(u_1, u_2)$ and $\epsilon > 0$. Let $U_1^n(m_1), m_1 \in [1, ..., 2^{nr_1}]$, be pairwise independent random sequences, each distributed according to $\prod_{i=1} P_{U_1}(u_{1,i})$. Similarly, Let $U_2^n(m_2), m_2 \in [1, ..., 2^{nr_2}]$, be pairwise independent random sequences, each distributed according to $\prod_{i=1} P_{U_2}(u_{2,i})$. Assume that $U_1^n(m_1) : m_1 \in [1, ..., 2^{nr_1}]$ and $U_2^n(m_2) : m_2 \in [1, ..., 2^{nr_2}]$ are independent.

Then, there exists $\delta(\epsilon)$ that goes to 0 as $\epsilon \to 0$ such that if

$$r_1 + r_2 > I(U_1; U_2) + \delta(\epsilon), \quad (18)$$

then

$$\lim_{n=\infty} \Pr\{(U^n 1(m_1), U_2^n(m_1)) \notin T_\epsilon^{(n)}(U_1, U_2) \, \forall m_1 \in [1, ..., 2^{nr_1}], m_2 \in [1, ..., 2^{nr_2}]\} = 0 \quad (19)$$

In addition to the prove, please explain, how it extends the covering lemma.