Multi User Information Theory

Homework 2
Capacity of the semi-deterministic relay channel

1) Semi-Deterministic Relay Channel.
Consider a relay channel as introduced in the class that is given by the conditional probability \( P(y, y_1|x, x_1) \). We have learned a coding scheme called partial decode and forward. This scheme turned to be optimal for a semi-deterministic relay channel, where \( Y_1 \) is a function of \((X, X_1)\), i.e., \( y_1 = f(x, x_1) \). Hence, the joint distribution of the semi-deterministic relay channel is of the form \( P(y, y_1|x, x_1) = P(y|x, x_1)1_{y_1=f(x, x_1)} \) where \( 1_{y_1=f(x, x_1)} \) is 1 if \( y_1 = f(x, x_1) \) and zero otherwise.

a) Show that any rate satisfying
\[
R \leq \min(H(Y_1|X_1) + I(X; Y|X_1, Y_1), I(X, X_1; Y))
\]
for some \( P(x, x_1) \) is achievable.

b) Show that a rate that is achievable must satisfy (1) for some joint distribution \( P(x, x_1) \).

c) Consider the example in Fig. 1. The relay observes \( X \) and encode it with a delay as in the regular relay setting. The input to the channel is \((X_i(m), X_1, i(Y_i−1))\). The output channel at time \( i \) i.e., \( Y_i \), is randomly chosen with equal probability to be either \( X_i \) or \( X_{1,i} \). Find the capacity of the example in Fig. 1. If analytical solution is not possible, you may use the computer to obtain a numerical solution.

d) Now, consider the case where the relay is taking an action \( A_i \). The role of the action is to decide if the relay observes or not observe \( X_i \). Namely, if \( A_i = 1 \) the relay observe \( X_i \) and if \( A_i = 0 \) the relay does not observe \( X_i \). There exists a constraint on the actions such that
\[
\frac{1}{n} \sum_{i=1}^{n} E[A_i] \leq \Gamma.
\]
constant. Similar to $X_{1,i}$ which may depend on $Y_{1}^{i-1}$ the action $A_i$ may also depend on $Y_{1}^{i-1}$. Fig. 2 depicts the setting. There exists a constraint on the relay action that $\frac{1}{n} \sum_{i=1}^{n} E[A_i] \leq \Gamma$.

Define the code of the setting and find the capacity region as a function of $\Gamma$. (Hint: Note that $A_i$ has a similar role as $X_{1,i}$; hence can you formulate the setting of Fig. 2 as being a regular formulation of semi-deterministic relay channel?)

e) Find the capacity region (numerically or analytically) where $\Gamma = 0$ and explain the result.

f) Draw using a computer the capacity region as a function of $\Gamma$.

Good Luck!!!