**Homework Set #3**

Gaussian Random variables and the Gaussian Relay

1. **Scaling.** Let \( h(X) = -\int f(x) \log f(x) \, dx \). Show \( h(AX) = \log | \text{det}(A) | + h(X) \).

2. **Concavity of determinants.** Let \( K_1 \) and \( K_2 \) be two symmetric non-negative definite \( n \times n \) matrices. Prove the result of Ky Fan [?]:

\[
| \lambda K_1 + (1-\lambda) K_2 | \geq | K_1 |^{\lambda} | K_2 |^{1-\lambda},
\]

for \( 0 \leq \lambda \leq 1 \), \( \lambda = 1 - \lambda \),

where \( | K | \) denotes the determinant of \( K \).

*Hint:* Let \( Z = X_{\theta} \), where \( X_1 \sim N(0, K_1) \), \( X_2 \sim N(0, K_2) \) and \( \theta \) = Bernoulli(\( \lambda \)). Then use \( h(Z | \theta) \leq h(Z) \).

3. **Bound on MMSE**

Given side information \( Y \) and estimator \( \hat{X}(Y) \), show that

\[
E[(X - \hat{X}(Y))^2] \geq \frac{1}{2\pi e} e^{2h(X|Y)}.
\]

4. **Gaussian mutual information.** Suppose that \( (X, Y, Z) \) are jointly Gaussian and that \( X \rightarrow Y \rightarrow Z \) forms a Markov chain. Let \( X \) and \( Y \) have correlation coefficient \( \rho_1 \) and let \( Y \) and \( Z \) have correlation coefficient \( \rho_2 \). Find \( I(X; Z) \).

5. **Gaussian Relay, decode and forward**

(a) For the Gaussian Relay that we have seen in class, derive the expression of the achievable region one by one use the decode and forward coding scheme.

(b) Draw the rate as a function of the parameters of the channel (each time draw as a function of one parameter), and compare it to the cut-set bound.