1st Semester 2011/12

Homework Set #3

Gaussian Random variables and the Gaussian Relay

1. **Scaling.** Let $h(\mathbf{X}) = -\int f(\mathbf{x}) \log f(\mathbf{x})\, d\mathbf{x}$. Show $h(A\mathbf{X}) = \log \mid \det(A) \mid + h(\mathbf{X})$.

2. **Concavity of determinants.** Let K_1 and K_2 be two symmetric non-negative definite $n \times n$ matrices. Prove the result of Ky Fan [**?**]:

$$\mid \lambda K_1 + \overline{\lambda} K_2 \mid \geq \mid K_1 \mid^{\lambda} \mid K_2 \mid^{\overline{\lambda}}, \quad \text{for } 0 \leq \lambda \leq 1,\ \overline{\lambda} = 1 - \lambda,$$

   where $\mid K \mid$ denotes the determinant of K.

   Hint: Let $\mathbf{Z} = \mathbf{X}_\theta$, where $\mathbf{X}_1 \sim N(0, K_1)$, $\mathbf{X}_2 \sim N(0, K_2)$ and $\theta =$ Bernoulli(λ). Then use $h(\mathbf{Z} \mid \theta) \leq h(\mathbf{Z})$.

3. **Bound on MMSE**

   Given side information Y and estimator $\hat{X}(Y)$, show that

$$E[(X - \hat{X}(Y))^2] \geq \frac{1}{2\pi e} e^{2h(X|Y)}.$$

4. **Gaussian mutual information**. Suppose that (X, Y, Z) are jointly Gaussian and that $X \to Y \to Z$ forms a Markov chain. Let X and Y have correlation coefficient ρ_1 and let Y and Z have correlation coefficient ρ_2. Find $I(X; Z)$.

5. **Gaussian Relay, decode and forward**

   (a) For the Gaussian Relay that we have seen in class, derive the expression of the achievable region one one use the decode and forward coding scheme.

   (b) Draw the rate as a function of the parameters of the channel (each time draw as a function of one parameter), and compare it to the cut-set bound.