Finite State Channels With and Without Feedback via Directed Information

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Information Theory and Applications
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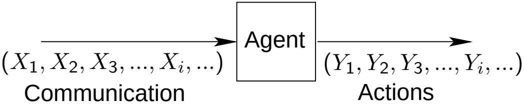
Measuring Causality

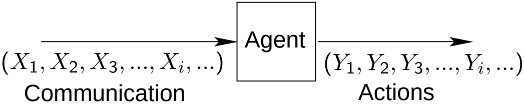


Measuring Causality

<u>Question</u>: Are the actions Y^i *caused* by the communication X^i?

not caused if $P(y_i|y^{i-1}, x^i) = P(y_i|y^{i-1})$, e.g., Y^i independent X^i.

caused if $P(y_i|y^{i-1}, x^i) \neq P(y_i|y^{i-1})$, e.g., $Y_i = X_i$ & $X_i \sim$ i.i.d.



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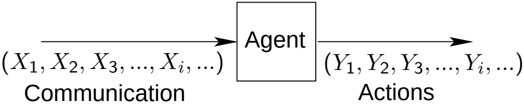
Does the mutual information $I(X^n; Y^n)$ measure causality?



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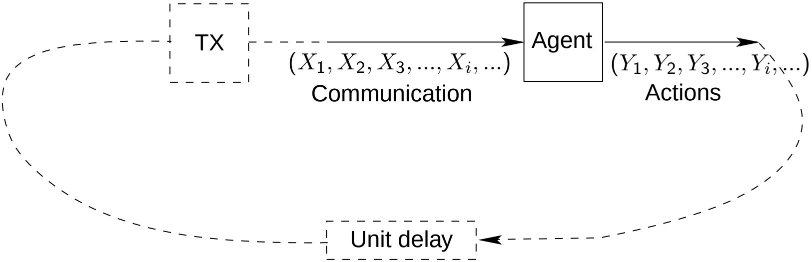

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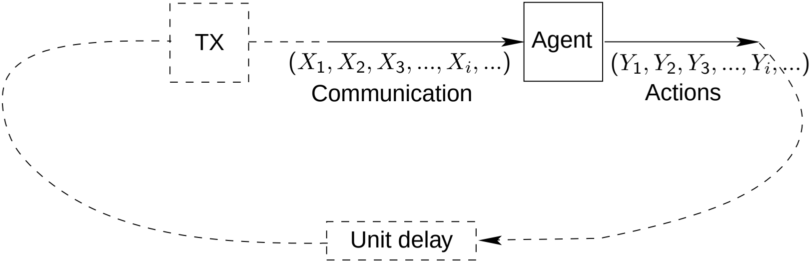
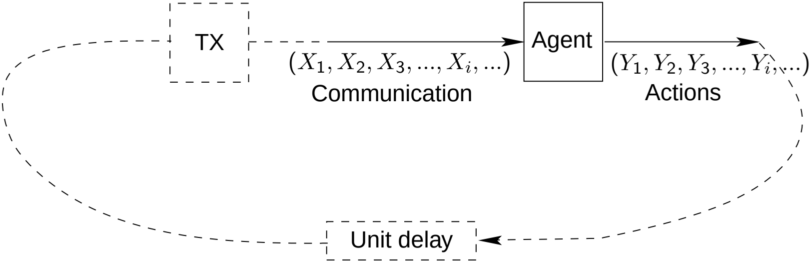


Example: Y_i i.i.d Bernoulli($\frac{1}{2}$) and $X_i = Y_{i-1}$.

$$\frac{1}{n} I(X^n; Y^n) = \frac{n-1}{n} \longrightarrow 1$$

Measuring Causality

Does the mutual information $I(X^n;Y^n)$ measure causality? No.



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Directed Information and Causal Conditioning

Directed Information [Marko73, Massey90]

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\begin{aligned}
I(X^n \to Y^n) &\triangleq H(Y^n) - H(Y^n||X^n) \\
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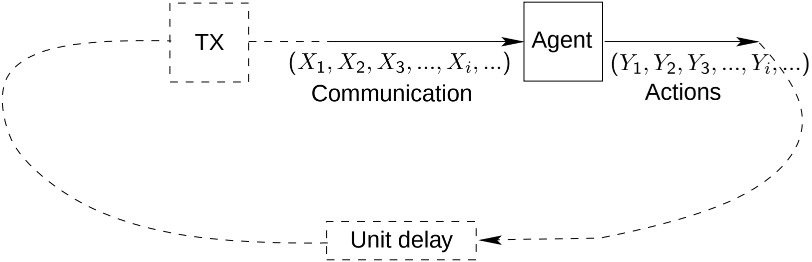
Measuring Causality

Consider $I(X^n \to Y^n)$ for measuring causality.



Example: Y_i i.i.d Bernoulli($\frac{1}{2}$) and $X_i = Y_{i-1}$.

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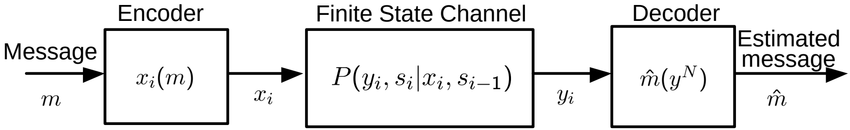
\begin{aligned}
I(X^n \to Y^n) &= H(Y^n) - H(Y^n||X^n) \\
&= \sum_{i=1}^{n} H(Y_i|Y^{i-1}) - H(Y_i|Y^{i-1}, X^i) = 0
\end{aligned}
$$

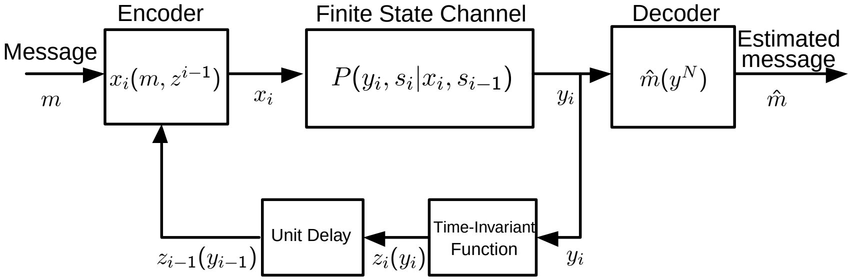
Finite State Channels



Finite State Channel(FSC) property:

$$P(y_i, s_i|x^i, s^{i-1}, y^{i-1}) = P(y_i, s_i|x_i, s_{i-1})$$

Finite State Channels with and without feedback



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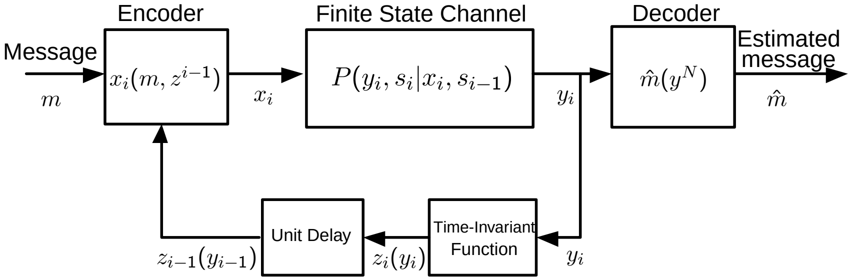

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Question: What is the channel capacity in this setting?

Answer

Theorem

For any FSC without feedback *[Gallager68]*

$$C_{FB} \geq \frac{1}{n} \max_{P(x^n)} \min_{s_0} I(X^n;Y^n|s_0) - \frac{\log|\mathcal{S}|}{n}$$

$$C_{FB} \leq \frac{1}{n} \max_{P(x^n)} \max_{s_0} I(X^n;Y^n|s_0) + \frac{\log|\mathcal{S}|}{n}$$

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Theorem

For any FSC with feedback *[P.&Weissman&Goldsmith06]*

$$C_{FB} \geq \frac{1}{n} \max_{P(x^n || z^n)} \min_{s_0} I(X^n \to Y^n | s_0) - \frac{\log |\mathcal{S}|}{n}$$

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Key properties

$$P(x^n, y^n) = P(x^n || y^{n-1}) P(y^n || x^n)$$

$$P(x^n, y^n) = P(x^n) P(y^n | x^n)$$

Key properties

$$P(x^n, y^n) = P(x^n||y^{n-1})P(y^n||x^n)$$

$$P(x^n, y^n) = P(x^n)P(y^n|x^n)$$

$$\frac{1}{n}\left|I(X^n \to Y^n) - I(X^n \to Y^n|S)\right| \leq \frac{H(S)}{n}$$

$$\frac{1}{n}\left|I(X^n; Y^n) - I(X^n; Y^n|S)\right| \leq \frac{H(S)}{n}$$

Gallager's proof for the case of non-feedback

1. Randomly chosen codewords with distribution

$$P^*(x^n) = \arg\max_{P(x^n)} \min_{s_0} I(X^n;Y^n|s_0).$$

2. Maximum likelihood decoder :

$$m^* = \arg\max_{m} P(y^n|m) = \arg\max_{m} P(y^n|x^n(m)).$$

3. Show that if

$$R < \sum_{y^n}\sum_{x^n} P(x^n)\cdot P(y^n|x^n) \ln \frac{P(y^n|x^n)}{\sum_{x^n} P(x^n)P(y^n|x^n)},$$

then $P_e \to 0$.

Proof for the case of time-invariant feedback

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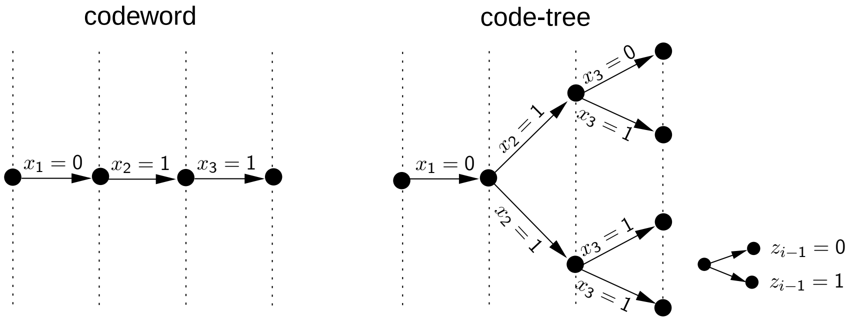

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Did we get directed information?

Recall first property:

$$P(x^n, y^n) = P(x^n||y^{n-1})P(y^n||x^n)$$

$$\sum_{x^n,y^n} P(x^n||y^{n-1}) \cdot P(y^n||x^n) \ln \frac{P(y^n||x^n)}{\sum_{x^n} P(x^n||y^{n-1}) \cdot P(y^n||x^n)} \stackrel{?}{=} I(X^n \to Y^n)$$

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Does always switching between $P(x^n) \leftrightarrow P(x^n||y^{n-1})$ and $P(y^n|x^n) \leftrightarrow P(y^n||x^n)$ work?

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Does always switching between $P(x^n) \leftrightarrow P(x^n||y^{n-1})$ and $P(y^n|x^n) \leftrightarrow P(y^n||x^n)$ work?
No! For instance we have

$$\sum_u P(y^n, u|x^n) = P(y^n|x^n)$$

but in general

$$\sum_u P(y^n, u||x^n) \neq P(y^n||x^n)$$

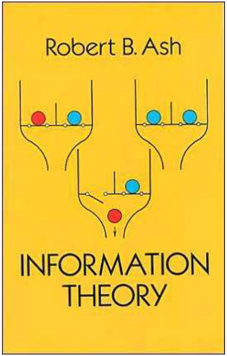
Capacity Results

For some cases the upper bound and the lower bound coincide.

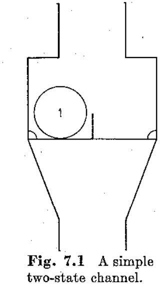
1. Initial state has positive probability for all states.
2. Compound channel setting (infinitely many FSCs $P_\theta(y_i, s_i|x_i, s_{i-1})$, $\theta \in \Theta$).
3. Indecomposable FSC and no ISI.
4. The state is computable at the the encoder, and all states are connected.

The trapdoor channel

Introduced by David Blackwell in 1961. [Ash65], [Ahlswede & Kaspi 87], [Ahlswede 98], [Kobayashi 02].

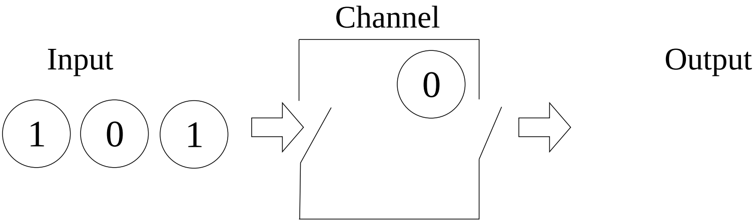


(a) Ash book



(b) D. Blackwell

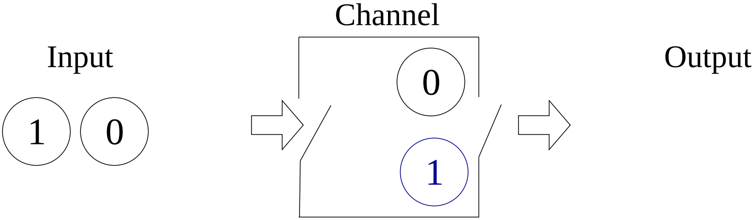
The Trapdoor Channel



$$s_t = s_{t-1} + x_t - y_t$$

$s_0 = 0$

The Trapdoor Channel



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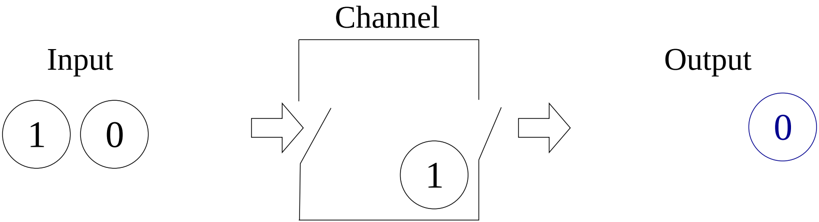
The Trapdoor Channel



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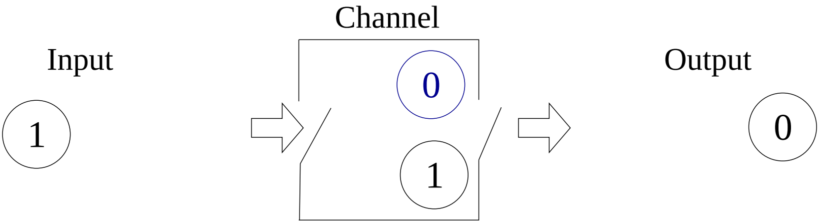
The Trapdoor Channel



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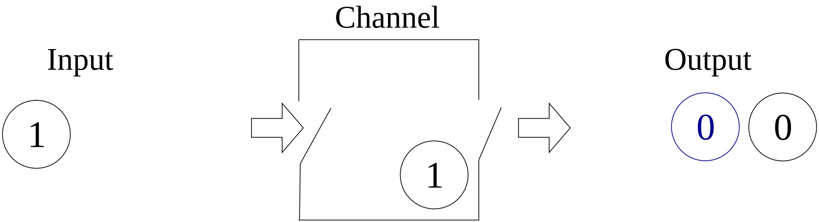

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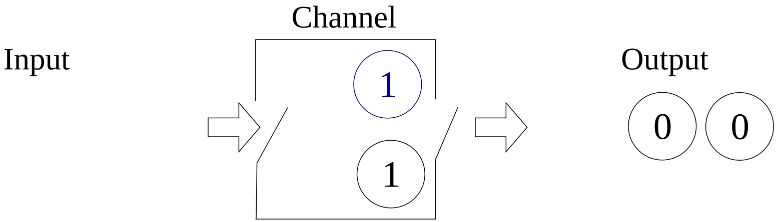

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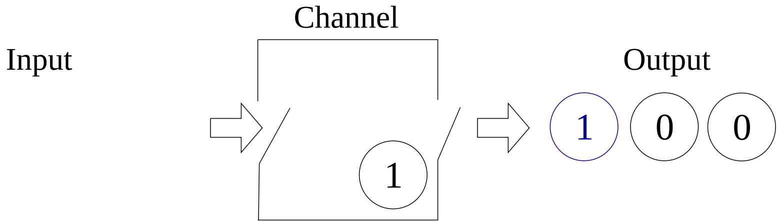

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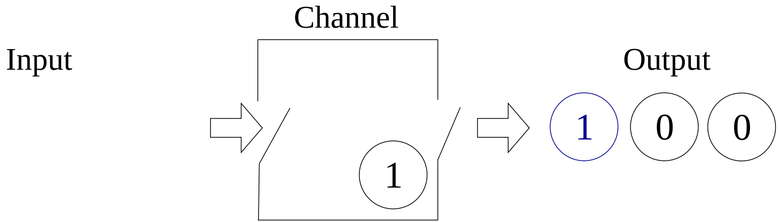
$x_3 = 1,\ s_3 = 1,\ y_3 = 1.$

The Trapdoor Channel



$$s_t = s_{t-1} + x_t - y_t$$

$s_0 = 0$
$x_1 = 1$, $s_1 = 1$, $y_1 = 0$,
$x_2 = 0$, $s_2 = 1$, $y_2 = 0$,
$x_3 = 1$, $s_3 = 1$, $y_3 = 1$.

Biochemical Interpretation [Berger 71]

Solving the trapdoor channel

- Initial state doesn't matter; upper and lower bounds become equal.

$$C_{FB} = \lim_{N\to\infty} \frac{1}{N} \max_{P(x^N||y^{N-1})} I(X^N \to Y^N)$$

- Converting it into a dynamic program. [Yang, Kavčić and Tatikonda05].
- Use value iteration algorithm to solve numerically the dynamic program.
- Verify the optimal solution through Bellman equation.

Solving the dynamic programming

Executed 20 value iterations: $J_{k+1} = T \circ J_k$

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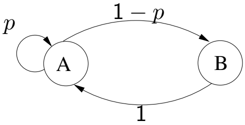
Entropy rate. Find the maximum entropy rate of the following two-state Markov chain:



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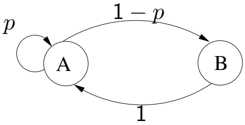

Solution: $H(\mathcal{X}) = \log \phi = 0.6942...$ bits (Golden Ratio: $\phi = \frac{\sqrt{5}+1}{2}$)

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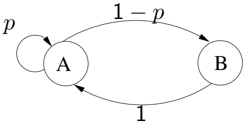

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Dynamic programming- Bellman equation

Theorem

(Bellman Equation.) If there exists a function $J(\beta)$ and a constant ρ that satisfy

$$J(\beta) = T \circ J(\beta) - \rho$$

then ρ is the optimal infinite horizon average reward.

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then ρ is the optimal infinite horizon average reward.

By constructing $J(\beta)$ and $\rho = \log \frac{\sqrt{5}+1}{2}$ that satisfies the Bellman equation we conclude that

$$C_{fb} = \log \frac{\sqrt{5}+1}{2}.$$

A scheme that achieves capacity

Let r^n denotes a sequences of length n with no two consecutive 1's.

$$00101010100101\ldots$$

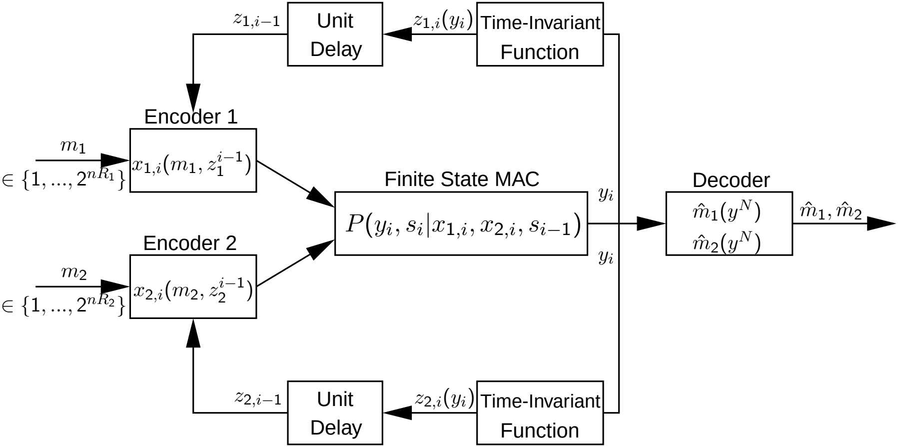
A simple scheme

Encoder: Map each message m to a sequence $[r^n(m)]$.
Decoder: The decoder decodes the sequence backward!

$$r_i = 1 \qquad \Rightarrow \qquad r_{i-1} = 0$$
$$r_i = 0 \qquad \Rightarrow \qquad r_{i-1} = y_i \oplus y_{i-1}$$

[P.&Cuff&Van-Roy&Weissman06]

MAC with time-invariant feedback



Finite State MAC

Let

$$\underline{\mathcal{R}}_n = \bigcup \begin{cases} R_1 \leq \min_{s_0} \frac{1}{n} I(X_1^n \to Y^n || X_2^n, s_0) - \frac{\log|\mathcal{S}|}{n}, \\ R_2 \leq \min_{s_0} \frac{1}{n} I(X_2^n \to Y^n || X_1^n, s_0) - \frac{\log|\mathcal{S}|}{n}, \\ R_1 + R_2 \leq \min_{s_0} \frac{1}{n} I((X_1, X_2)^n \to Y^n | s_0) - \frac{\log|\mathcal{S}|}{n}, \end{cases}$$

the union is over input distribution $P(x_1^n || z_1^{n-1}) P(x_2^n || z_2^{n-1})$.

Theorem

For any FS-MAC with time invariant feedback, $\underline{\mathcal{R}}_n$ is an inner bound.
The outer bound is given in terms of limit. P.&Weissman&Chen07

An Ingredient of the Proof: Sup-additivity

Let a_n be a bounded sequence of real numbers.
If na_n is sup-additive, i.e., for all $n > k$

$$na_n \geq ka_k + (n-k)a_{n-k},$$

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An Ingredient of the Proof: Sup-additivity

Let A_n be a bounded sequence of 2D sets.
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then

$$\lim_{n\to\infty} A_n = \bigcup_{n\geq 1} A_n.$$

An Ingredient of the Proof: Sub-additivity

Let A_n be a bounded sequence of 2D convex sets.
If nA_n is sub-additive, i.e., for all $n > k$

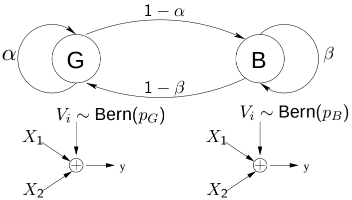
$$
\begin{aligned}
nA_n &\subseteq kA_k + (n-k)A_{n-k}, \\
A_n &\subseteq \frac{k}{n}A_k + \frac{(n-k)}{n}A_{n-k},
\end{aligned}
$$

then

$$\lim_{n\to\infty} A_n = \bigcap_{n\geq 1} A_n.$$

Applications

- $$\mathcal{R} = 0 \iff \mathcal{R}_{fb} = 0$$
  The proof is based on the fact that
  $$\max_{Q(x^n||y^{n-1})} I(X^n \to Y^n) = 0 \iff \max_{Q(x^n)} I(X^n; Y^n) = 0$$
- Gillbert-Elliot MAC.



  - feedback does not increase capacity.
  - source-channel separation theorem holds for the lossless case.

Portfolio Theory

Consider a horse-race market

- X_i - the horse that wins at time i.
- Y_i - side information available at time i.

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(X_i, Y_i) general processes [P.&Kim&Weissman08]

The optimal strategy is to invest the capital proportional to $P(x_i|x^{i-1}, y^i)$. The increase in the growth rate due to side information is

$$\Delta W = I(Y^n \to X^n).$$

Summary

Intuition

$I(X^n; Y^n)$ amount of uncertainty about Y^n reduced by knowing X^n

$I(X^n \to Y^n)$ amount of uncertainty about Y^n reduced by knowing X^n causally.

Results

- capacity of point-to-point FSC (e.g., trapdoor channel)
- capacity of FS-MAC (e.g., Gilbert-Elliot MAC)
- the increase in growth rate due to side information

Other results with directed information

[Marko73]
[Kramer98]
[Chen/Berger05]
[Tatikonda/Mitter07]
[Venkataramanan/Pradhan07]

[Massey90]
[Tatikonda00]
[Yang/Kavcic/Tatikonda05]
[Kim07]

Thank You

Academic advisors
Tsachy Weissman
Tom Cover

Collaborators
Jun Chen, Paul Cuff, Young-Han Kim, Andrea Goldsmith,
Brooke Shrader, Benjamin Van-Roy.

Thank You

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Thank you for attending the talk!