Mathematical methods in communication

Lecture BEC Capacity

Lecturer: *Scribe:*

I. Binary Erasure Channel (BEC)

A Binary Erasure Channel (BEC) is a common communications channel model used in coding theory and information theory. In this model, a transmitter sends a bit (a zero or a one), and the receiver either receives the bit or it receives a symbol '?' that represents that the bit was erased, namely, the receiver knows that a bit was sent but it does not know which one. For instance, in an internet protocol, '?' may represent that the packet received was corrupted and $\{0, 1\}$ may represent tow possible packets.

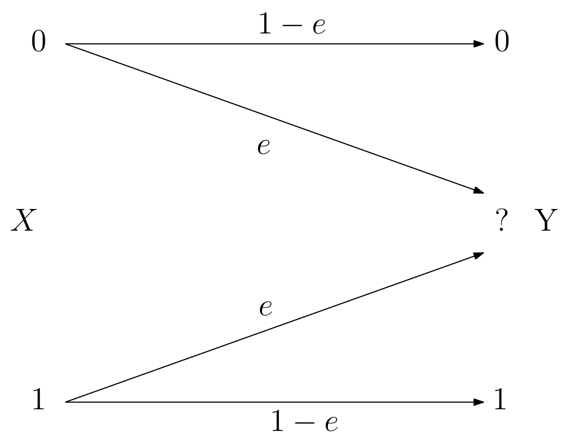

Fig. 1. Binary Erasure Channel

Where '?' stands for an erased bit.

II. Feedback BEC Channel

Assume the transmitter at time i knows the previous outputs of the channel,i.e., y^{i-1}, so we can re-transmit the "erased" bit. In order to successfully receive n bits, we must transmit $\frac{n}{1-e}$ bits: First we transmit n bits. Since $P_e = e$ and we have feedback, we know that $e \cdot n$ bits were erased, so we need to re-transmit them. This time, $e \cdot en$ bits were erased, so once again we re-transmit them, and so on. All together, we have transmitted:

$$n + en + e^2 n + \cdots = \sum_{j=0}^{\infty} e^j n = n \cdot \sum_{j=0}^{\infty} e^j = n \cdot \frac{1}{1-e} = \frac{n}{1-e} \tag{1}$$

In order to successfully receive n bits.

Definition 1 (Code Rate) In telecommunication and information theory, the code rate R of a channel code is the proportion of the data-stream that is useful (non-redundant). That is, if the code rate is $R = k/n$, for every k bits of useful information, the coder generates totally n bits of data, of which n-k are redundant.
In the feedback erasure channel scenario, the ratio of useful information to total sent information was $R = \frac{n}{n/(1-e)} = 1 - e$

III. No-Feedback channel

Since in this case we have less information than that given in the feedback-channel scenario, we can expect a rate lower than $R_{feedback}$, meaning that the upper bound on the achievable rate in this case is expected to be (lower than or equal to) $R^* = 1 - e$.

The capacity of the channel is given by $C = Sup_{P_X} I(X;Y)$, and C is known to be the upper bound of the achievable rates. So, we try to find the channel capacity:

$$I(X;Y) = H(X) - H(X|Y) \tag{2}$$

$$= H(X) - \sum_{\psi \in Y} P(y = \psi) H(X|y = \psi) \tag{3}$$

$$= H(X) - P(y = 0)H(X|y = 0) - P(y = 1)H(X|y = 1) - P(y =' ?')H(X|y =' ?') \tag{4}$$

Where '?' stands for an erased bit (See Figure I).
Since the model of the channel suggests that for a successfully received bit we know the sent bit with probability of 1, meaning X is determined by Y (given $y \neq' ?'$), we get $H(X|y = 0) = H(X|y = 1) = 0$. Also, if the bit was erased, by the symmetry of the channel we have no additional information regarding the value of the transmitted bit. Therefore $H(X|Y =' ?') = H(X)$, and $P(y =' ?') = e$ by symmetry. Substituting this into Eq. 2:

$$I(X;Y) = H(X) - P(y =' ?')H(X|y =' ?') = H(X) - e \cdot H(X) = (1 - e)H(X) \tag{5}$$

Finally, we need to find the supremum of the mutual information over all the possible distributions of X.

$$C = \sup_{P_X} I(X;Y) = (1 - e) \sup_{P_X} H(X) = 1 - e \tag{6}$$

Where the last equation holds for $X \sim Bernoulli(\frac{1}{2})$.Therefore an upper bound on the achievable rate is indeed $R \leq C = 1 - e$, as suggested.