I. Binary Erasure Channel (BEC)

A Binary Erasure Channel (BEC) is a common communications channel model used in coding theory and information theory. In this model, a transmitter sends a bit (a zero or a one), and the receiver either receives the bit or it receives a symbol ‘?’ that represents that the bit was erased, namely, the receiver knows that a bit was sent but it does not know which one. For instance, in an internet protocol, '?' may represent that the packet received was corrupted and \{0, 1\} may represent two possible packets.

\[ \begin{array}{c}
0 & \overset{1-e}{\rightarrow} & 0 \\
\downarrow e & & \uparrow e \\
X & \overset{?}{\rightarrow} & Y \\
\downarrow e & & \uparrow e \\
1 & \overset{1-e}{\rightarrow} & 1
\end{array} \]

Fig. 1. Binary Erasure Channel

Where '?' stands for an erased bit.

II. Feedback BEC Channel

Assume the transmitter at time \( i \) knows the previous outputs of the channel, i.e., \( y^{i-1} \), so we can re-transmit the “erased” bit. In order to successfully receive \( n \) bits, we must transmit \( \frac{n}{1-e} \) bits: First we transmit \( n \) bits. Since \( P_e = e \) and we have feedback, we know that \( e \cdot n \) bits were erased, so we need to re-transmit them. This time, \( e \cdot en \) bits were erased, so once again we re-transmit them, and so on. All together, we have transmitted:

\[
\sum_{j=0}^{\infty} e^j n = n \cdot \frac{1}{1-e} = \frac{n}{1-e}
\]
In order to successfully receive $n$ bits.

**Definition 1 (Code Rate)** In telecommunication and information theory, the code rate $R$ of a channel code is the proportion of the data-stream that is useful (non-redundant). That is, if the code rate is $R = k/n$, for every $k$ bits of useful information, the coder generates totally $n$ bits of data, of which $n-k$ are redundant.

In the feedback erasure channel scenario, the ratio of useful information to total sent information was $R = \frac{n}{n/(1-e)} = 1 - e$

### III. No-Feedback channel

Since in this case we have less information than that given in the feedback-channel scenario, we can expect a rate lower than $R_{\text{feedback}}$, meaning that the upper bound on the achievable rate in this case is expected to be (lower than or equal to) $R^* = 1 - e$.

The capacity of the channel is given by $C = \text{Sup}_{P_X} I(X;Y)$, and $C$ is known to be the upper bound of the achievable rates. So, we try to find the channel capacity:

\[
I(X;Y) = H(X) - H(X|Y) = H(X) - \sum_{\psi \in Y} P(y = \psi)H(X|y = \psi) \\
= H(X) - P(y = 0)H(X|y = 0) - P(y = 1)H(X|y = 1) - P(y ='?)H(X|y ='?)
\]

Where '?' stands for an erased bit (See Figure I).

Since the model of the channel suggests that for a successfully received bit we know the sent bit with probability of 1, meaning $X$ is determined by $Y$ (given $y \neq'?'$), we get $H(X|y = 0) = H(X|y = 1) = 0$. Also, if the bit was erased, by the symmetry of the channel we have no additional information regarding the value of the transmitted bit. Therefore $H(X|Y ='?) = H(X)$, and $P(y ='?) = e$ by symmetry. Substituting this into Eq. 2:

\[
I(X;Y) = H(X) - P(y ='?)H(X|y ='?) = H(X) - e \cdot H(X) = (1 - e)H(X)
\]

Finally, we need to find the supremum of the mutual information over all the possible distributions of $X$.

\[
C = \text{sup}_{P_X} I(X;Y) = (1 - e) \text{sup}_{P_X} H(X) = 1 - e
\]

Where the last equation holds for $X \sim \text{Bernoulli}(\frac{1}{2})$. Therefore an upper bound on the achievable rate is indeed $R \leq C = 1 - e$, as suggested.