Homework Set #5
Polar codes

1. **Polarization and the idea of polar codes (28 Points)**: The question is about polarization effect in memoryless channels that can lead to simple coding schemes that achieve the capacity which are called polar codes.

   (a) Consider the channel in Fig. 1 where two parallel binary erasure channels can be used at once (the input is $X = (X_1, X_2)$). The inputs alphabets are binary, so that Y_1 and

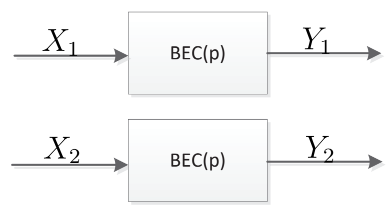

Figure 1: Two parallel binary erasure channels

   Y_2 are the outputs of a BEC(p) with inputs X_1 and X_2, respectively. Compute the capacity of this channel, namely,

$$\max_{p(x_1,x_2)} I(X_1, X_2; Y_1, Y_2). \tag{1}$$

   What is the input distribution $p(x_1, x_2)$ that achieves the capacity?

   (b) Consider the system in Fig. 2, where addition is modulo 2:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}$$

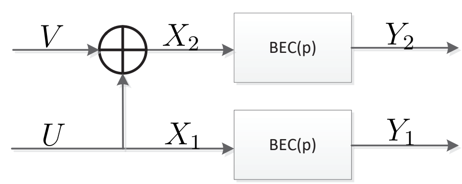

Figure 2: Two parallel binary erasure channels with modified inputs

Compute the capacity of the new channel, i.e. $\max_{p(u,v)} I(U,V;Y_1,Y_2)$.

What is the $p(u,v)$ that achieves the capacity?

Next, the channel is decomposed into two parallel channels as appears in Fig. 3. The input of Channel 1 is U and its output is (Y_1, Y_2, V). The input of Channel 2 is V and its output is (Y_1, Y_2).

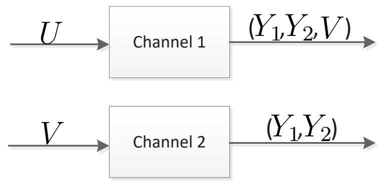

Figure 3: Two new channels

(c) Compute the expressions $I(U;Y_1,Y_2,V)$ and $I(V;Y_1,Y_2)$ with respect to the $p(u,v)$ that achieves the maximum in (b). What is the sum of the expressions you computed?

(d) Compare the mutual information of Channels 1 and 2 with the capacity of a binary erasure channel (that is, write $<$, $>$ or $=$ with simple proof).

*For large n, repeating this decomposition n times, ends up in nc clean channels and in $n(1-c)$ totally noisy channels. This is the main idea of polar codes, which achieves capacity.