Homework Set #3  
Rates definitions, Channel Coding, Source-Channel coding

1. Rates
   
   (a) **Channels coding Rate:** Assuming you are sending 1024 different messages using 20 usages of a channel. What is the rate (in bits per channel use) that you send.
   
   (b) **Source coding Rate:** Assuming you have a file with $10^6$ Ascii characters, where the alphabet of Ascii characters is 256. Assume that each Ascii character is represented by bits (binary alphabet before compression). After compressing it we get $4 \times 10^6$ bits. What is the compression rate?

2. Preprocessing the output.
   
   One is given a communication channel with transition probabilities $p(y \mid x)$ and channel capacity $C = \max_{p(x)} I(X;Y)$. A helpful statistician preprocesses the output by forming $\hat{Y} = g(Y)$, yielding a channel $p(\hat{y} \mid x)$. He claims that this will strictly improve the capacity.
   
   (a) Show that he is wrong.
   
   (b) Under what conditions does he not strictly decrease the capacity?

3. The Z channel.
   
   The Z-channel has binary input and output alphabets and transition probabilities $p(y \mid x)$ given by the following matrix:
   $$Q = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} \quad x, y \in \{0, 1\}$$
   
   Find the capacity of the Z-channel and the maximizing input probability distribution.

4. Using two channels at once.
   
   Consider two discrete memoryless channels $(X_1, p(y_1 \mid x_1), Y_1)$ and $(X_2, p(y_2 \mid x_2), Y_2)$ with capacities $C_1$ and $C_2$ respectively. A new channel $(X_1 \times X_2, p(y_1 \mid x_1) \times p(y_2 \mid x_2), Y_1 \times Y_2)$ is formed in which $x_1 \in X_1$ and $x_2 \in X_2$, are simultaneously sent, resulting in $y_1, y_2$. Find the capacity of this channel.
5. **A channel with two independent looks at Y.**

Let $Y_1$ and $Y_2$ be conditionally independent and conditionally identically distributed given $X$. Thus $p(y_1, y_2|x) = p(y_1|x)p(y_2|x)$.

(a) Show $I(X; Y_1, Y_2) = 2I(X; Y_1) - I(Y_1; Y_2)$.

(b) Conclude that the capacity of the channel

\[
\begin{array}{c}
X \\
\end{array}
\begin{array}{c}
\text{square} \\
\end{array}
\begin{array}{c}
(Y_1, Y_2) \\
\end{array}
\]

is less than twice the capacity of the channel

\[
\begin{array}{c}
X \\
\end{array}
\begin{array}{c}
\text{square} \\
\end{array}
\begin{array}{c}
Y_1 \\
\end{array}
\]

6. **Choice of channels.**

Find the capacity $C$ of the union of 2 channels $(X_1, p_1(y_1|x_1), Y_1)$ and $(X_2, p_2(y_2|x_2), Y_2)$ where, at each time, one can send a symbol over channel 1 or over channel 2 but not both. Assume the output alphabets are distinct and do not intersect.

(a) Show $2^C = 2^{C_1} + 2^{C_2}$.

(b) What is the capacity of this Channel?

7. **Cascaded BSCs.**

Consider the two discrete memoryless channels $(X, p_1(y|x), Y)$ and $(Y, p_2(z|y), Z)$.

Let $p_1(y|x)$ and $p_2(z|y)$ be binary symmetric channels with crossover probabilities $\lambda_1$ and $\lambda_2$ respectively.
(a) What is the capacity $C_1$ of $p_1(y|x)$?

(b) What is the capacity $C_2$ of $p_2(z|y)$?

(c) We now cascade these channels. Thus $p_3(z|x) = \sum_y p_1(y|x)p_2(z|y)$.

What is the capacity $C_3$ of $p_3(z|x)$? Show $C_3 \leq \min\{C_1, C_2\}$.

(d) Now let us actively intervene between channels 1 and 2, rather than passively transmitting $y^n$. What is the capacity of channel 1 followed by channel 2 if you are allowed to decode the output $y^n$ of channel 1 and then reencode it as $\tilde{y}^n$ for transmission over channel 2? (Think $W \rightarrow x^n(W) \rightarrow y^n \rightarrow \tilde{y}^n(y^n) \rightarrow z^n \rightarrow \hat{W}$.)

(e) What is the capacity of the cascade in part c) if the receiver can view both $Y$ and $Z$?

8. Channel capacity

(a) What is the capacity of the following channel in Fig. 1 (appears on the next page).

(b) Provide a simple scheme that can transmit at rate $R = \log_2 3$ bits through this channel.
9. **A channel with a switch.** Consider the channel that is depicted in Fig. 2, there are two channels with the conditional probabilities \( p(y_1|x) \) and \( p(y_2|x) \). These two channels have common input alphabet \( X \) and two disjoint output alphabets \( Y_1, Y_2 \) (a symbol that appears in \( Y_1 \) can’t appear in \( Y_2 \)). The position of the switch is determined by R.V \( Z \) which is independent of \( X \), where \( \Pr(Z = 1) = \lambda \).

\( X \)
- \( p(y_1|x) \rightarrow Y_1 \)
- \( p(y_2|x) \rightarrow Y_2 \)

\( Z = 1 \) and \( Z = 2 \)

Figure 2: The channel.

(a) Show that
\[
I(X;Y) = \lambda I(X;Y_1) + \bar{\lambda} I(X;Y_2). \tag{1}
\]

(b) The capacity of this system is given by \( C = \max_{p(x)} I(X;Y) \). Show that
\[
C \leq \lambda C_1 + \bar{\lambda} C_2, \tag{2}
\]
where \( C_i = \max_{p(x)} I(X; Y_i) \). When is equality achieved?

(c) The sub-channels defined by \( p(y_1|x) \) and \( p(y_2|x) \) are now given in Fig.9, where \( p = \frac{1}{2} \). Find the input probability \( p(x) \) that maximizes \( I(X; Y) \). For this case, does the equality stand in Eq. (2)? explain!

Figure 3: (a) describes channel 1 - BSC with transition probability \( p \). (b) describes channel 2 - Z channel with transition probability \( p \).

10. Channel with state

A discrete memoryless (DM) state dependent channel with state space \( S \) is defined by an input alphabet \( \mathcal{X} \), an output alphabet \( \mathcal{Y} \) and a set of channel transition matrices \( \{p(y|x, s)\}_{s \in S} \). Namely, for each \( s \in S \) the transmitter sees a different channel. The capacity of such a channel where the state is know causally to both encoder and decoder is given by:

\[
C = \max_{p(x|s)} I(X; Y|S).
\] (3)

Let \( |S| = 3 \) and the three different channels (one for each state \( s \in S \)) are as illustrated in Fig.4

The state process is i.i.d. according to the distribution \( p(s) \).

(a) Find an expression for the capacity of the S-channel (the channel the transmitter sees given \( S = 1 \)) as a function of \( \epsilon \).

(b) Find an expression for the capacity of the BSC (the channel the transmitter sees given \( S = 2 \)) as a function of \( \delta \).
Figure 4: The three state dependent channel.

(c) Find an expression for the capacity of the Z-channel (the channel the transmitter sees given \( S = 3 \)) as a function of \( \epsilon \).

(d) Find an expression for the capacity of the DM state dependent channel (using formula (3)) for \( p(s) = \left[ \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{6} \right] \) as a function of \( \epsilon \) and \( \delta \).

(e) Let us define a conditional probability matrix \( P_{X|S} \) for two random variables \( X \) and \( S \) with \(|\mathcal{X}| = \{0, 1\}\) and \(|\mathcal{S}| = \{1, 2, 3\}\), by:

\[
[P_{X|S}]^{3,2}_{i=1,j=1} = p(x = j - 1 | s = i). \tag{4}
\]

What is the input conditional probability matrix \( P_{X|S} \) that achieves the capacity you have found in (d)?

11. Modulo Channel

(a) Consider the DMC defined as follows: Output \( Y = X \oplus_2 Z \) where \( X \), taking values in \{0, 1\}, is the channel input, \( \oplus_2 \) is the modulo-2 summation operation, and \( Z \) is binary channel noise uniform over \{0, 1\} and independent of \( X \). What is the capacity of this channel?

(b) Consider the channel of the previous part, but suppose that instead of modulo-2 addition \( Y = X \oplus_2 Z \), we perform modulo-3 addition \( Y = X \oplus_3 Z \). Now what is the capacity?
12. **Cascaded BSCs**: Given is a cascade of $k$ identical and independent binary symmetric channels, each with crossover probability $\alpha$.

(a) In the case where no encoding or decoding is allowed at the intermediate terminals, what is the capacity of this cascaded channel as a function of $k, \alpha$.

(b) Now, assume that encoding and decoding is allowed at the intermediate points, what is the capacity as a function of $k, \alpha$.

(c) What is the capacity of each of the above settings in the case where the number of cascaded channels, $k$, goes to infinity?