Homework Set #3
Rates definitions, Channel Coding, Source-Channel coding

1. **Rates**

   (a) **Channels coding Rate:** Assuming you are sending 1024 different messages using 20 usages of a channel. What is the rate (in bits per channel use) that you send.

   (b) **Source coding Rate:** Assuming you have a file with 10^6 Ascii characters, where the alphabet of Ascii characters is 256. Assume that each Ascii character is represented by bits (binary alphabet before compression). After compressing it we get $4 * 10^6$ bits. What is the compression rate?

2. **Preprocessing the output.**
   One is given a communication channel with transition probabilities $p(y \mid x)$ and channel capacity $C = \max_{p(x)} I(X;Y)$. A helpful statistician preprocesses the output by forming $\tilde{Y} = g(Y)$, yielding a channel $p(\tilde{y}|x)$. He claims that this will strictly improve the capacity.

   (a) Show that he is wrong.

   (b) Under what conditions does he not strictly decrease the capacity?

3. **The Z channel.**
   The Z-channel has binary input and output alphabets and transition probabilities $p(y|x)$ given by the following matrix:

$$Q = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} \qquad x, y \in \{0, 1\}$$

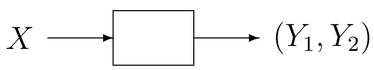
   Find the capacity of the Z-channel and the maximizing input probability distribution.

4. **Using two channels at once.**
   Consider two discrete memoryless channels $(\mathcal{X}_1, p(y_1 \mid x_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p(y_2 \mid x_2), \mathcal{Y}_2)$ with capacities C_1 and C_2 respectively. A new channel $(\mathcal{X}_1 \times \mathcal{X}_2, p(y_1 \mid x_1) \times p(y_2 \mid x_2), \mathcal{Y}_1 \times \mathcal{Y}_2)$ is formed in which $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$, are *simultaneously* sent, resulting in y_1, y_2. Find the capacity of this channel.

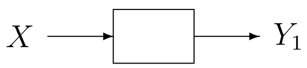
5. **A channel with two independent looks at Y.**
Let Y_1 and Y_2 be conditionally independent and conditionally identically distributed given X. Thus $p(y_1, y_2|x) = p(y_1|x)p(y_2|x)$.

(a) Show $I(X; Y_1, Y_2) = 2I(X; Y_1) - I(Y_1; Y_2)$.

(b) Conclude that the capacity of the channel

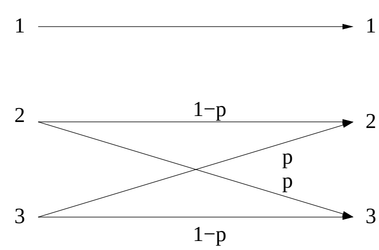


is less than twice the capacity of the channel



6. **Choice of channels.**
Find the capacity C of the union of 2 channels $(\mathcal{X}_1, p_1(y_1|x_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p_2(y_2|x_2), \mathcal{Y}_2)$ where, at each time, one can send a symbol over channel 1 or over channel 2 but not both. Assume the output alphabets are distinct and do not intersect.

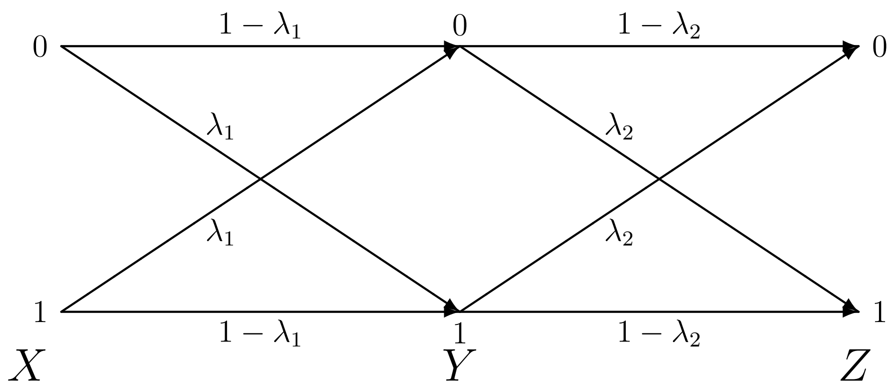
(a) Show $2^C = 2^{C_1} + 2^{C_2}$.



(b) What is the capacity of this Channel?

7. **Cascaded BSCs.**
Consider the two discrete memoryless channels $(\mathcal{X}, p_1(y|x), \mathcal{Y})$ and $(\mathcal{Y}, p_2(z|y), \mathcal{Z})$.

Let $p_1(y|x)$ and $p_2(z|y)$ be binary symmetric channels with crossover probabilities λ_1 and λ_2 respectively.



(a) What is the capacity C_1 of $p_1(y|x)$?

(b) What is the capacity C_2 of $p_2(z|y)$?

(c) We now cascade these channels. Thus $p_3(z|x) = \sum_y p_1(y|x)p_2(z|y)$. What is the capacity C_3 of $p_3(z|x)$? Show $C_3 \leq \min\{C_1, C_2\}$.

(d) Now let us actively intervene between channels 1 and 2, rather than passively transmitting y^n. What is the capacity of channel 1 followed by channel 2 if you are allowed to decode the output y^n of channel 1 and then reencode it as $\tilde{y}^n$ for transmission over channel 2? (Think $W \longrightarrow x^n(W) \longrightarrow y^n \longrightarrow \tilde{y}^n(y^n) \longrightarrow z^n \longrightarrow \hat{W}$.)

(e) What is the capacity of the cascade in part c) if the receiver can view *both* Y and Z?

8. **Channel capacity**

(a) What is the capacity of the following channel in Fig. 1 (appears on the next page).

(b) Provide a simple scheme that can transmit at rate $R = \log_2 3$ bits through this channel.

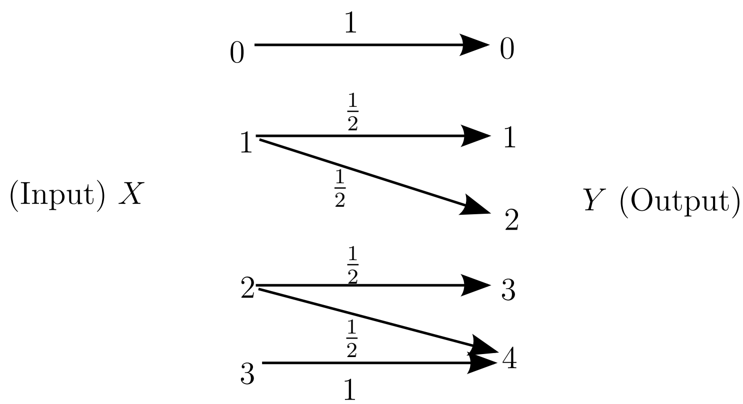


Figure 1: A channel

9. **A channel with a switch.** Consider the channel that is depicted in Fig.2, there are two channels with the conditional probabilities $p(y_1|x)$ and $p(y_2|x)$. These two channels have common input alphabet $\mathcal{X}$ and two **disjoint** output alphabets $\mathcal{Y}_1, \mathcal{Y}_2$ (a symbol that appears in $\mathcal{Y}_1$ can't appear in $\mathcal{Y}_2$). The position of the switch is determined by R.V Z which is independent of X, where $\Pr(Z = 1) = \lambda$.

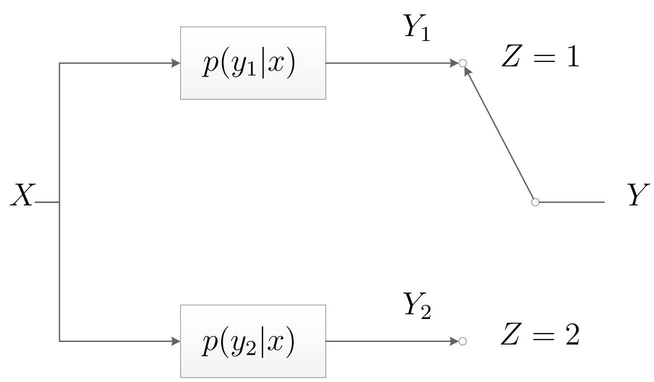


Figure 2: The channel.

(a) Show that

$$I(X;Y) = \lambda I(X;Y_1) + \bar{\lambda} I(X;Y_2). \tag{1}$$

(b) The capacity of this system is given by $C = \max_{p(x)} I(X;Y)$. Show that

$$C \leq \lambda C_1 + \bar{\lambda} C_2, \tag{2}$$

where $C_i = \max_{p(x)} I(X;Y_i)$.
When is equality achieved?

(c) The sub-channels defined by $p(y_1|x)$ and $p(y_2|x)$ are now given in Fig.9, where $p = \frac{1}{2}$.
Find the input probability $p(x)$ that maximizes $I(X;Y)$.
For this case, does the equality stand in Eq. (2)? explain!

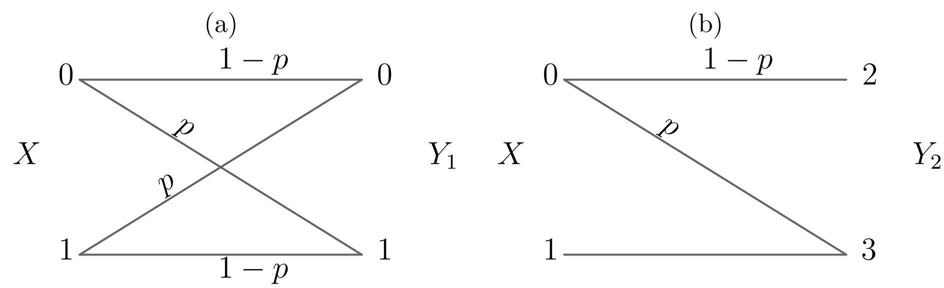


Figure 3: (a) describes channel 1 - BSC with transition probability p. (b) describes channel 2 - Z channel with transition probability p.

10. **Channel with state**

A discrete memoryless (DM) state dependent channel with state space $\mathcal{S}$ is defined by an input alphabet $\mathcal{X}$, an output alphabet $\mathcal{Y}$ and a set of channel transition matrices $\{p(y|x,s)\}_{s\in\mathcal{S}}$. Namely, for each $s \in \mathcal{S}$ the transmitter sees a different channel. The capacity of such a channel where the state is know causally to both encoder and decoder is given by:

$$C = \max_{p(x|s)} I(X;Y|S). \quad (3)$$

Let $|\mathcal{S}| = 3$ and the three different channels (one for each state $s \in \mathcal{S}$) are as illustrated in Fig.4

The state process is i.i.d. according to the distribution $p(s)$.

(a) Find an expression for the capacity of the S-channel (the channel the transmitter sees given $S = 1$) as a function of ϵ.

(b) Find an expression for the capacity of the BSC (the channel the transmitter sees given $S = 2$) as a function of δ.

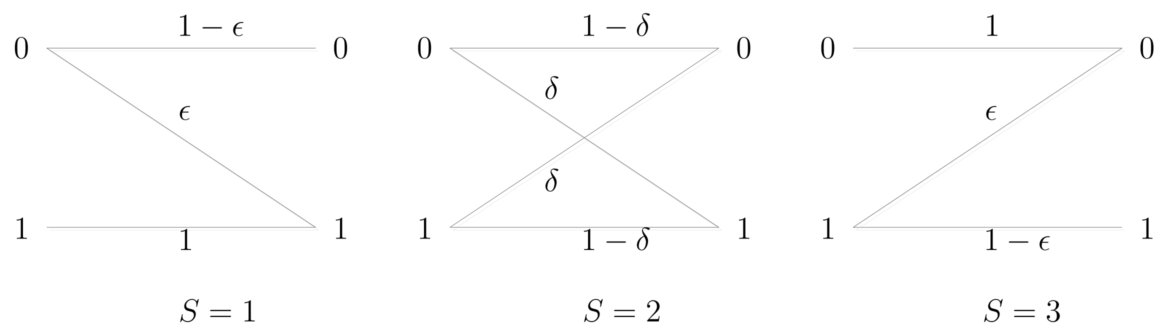


Figure 4: The three state dependent channel.

(c) Find an expression for the capacity of the Z-channel (the channel the transmitter sees given $S = 3$) as a function of ϵ.

(d) Find an expression for the capacity of the DM state dependent channel (using formula (3)) for $p(s) = [\frac{1}{2} \ \frac{1}{3} \ \frac{1}{6}]$ as a function of ϵ and δ.

(e) Let us define a conditional probability matrix $P_{X|S}$ for two random variables X and S with $|\mathcal{X}| = \{0, 1\}$ and $|\mathcal{S}| = \{1, 2, 3\}$, by:

$$\left[P_{X|S}\right]_{i=1,j=1}^{3,2} = p(x = j - 1 | s = i). \tag{4}$$

What is the input conditional probability matrix $P_{X|S}$ that achieves the capacity you have found in (d)?

11. **Modulo Channel**

(a) Consider the DMC defined as follows: Output $Y = X \oplus_2 Z$ where X, taking values in $\{0, 1\}$, is the channel input, $\oplus_2$ is the modulo-2 summation operation, and Z is binary channel noise uniform over $\{0, 1\}$ and independent of X. What is the capacity of this channel?

(b) Consider the channel of the previous part, but suppose that instead of modulo-2 addition $Y = X \oplus_2 Z$, we perform modulo-3 addition $Y = X \oplus_3 Z$. Now what is the capacity?

12. **Cascaded BSCs**: Given is a cascade of k identical and independent binary symmetric channels, each with crossover probability α.

   (a) In the case where no encoding or decoding is allowed at the intermediate terminals, what is the capacity of this cascaded channel as a function of k, α.

   (b) Now, assume that encoding and decoding is allowed at the intermediate points, what is the capacity as a function of k, α.

   (c) What is the capacity of each of the above settings in the case where the number of cascaded channels, k, goes to infinity?