Introduction to Information Theory and Machine Learning Aug 27th, 2020.
(Prof. Permuter Haim, Mr. Eli Shmuel, Mr. Dor Tsur and Mr. Ben Marinberg)

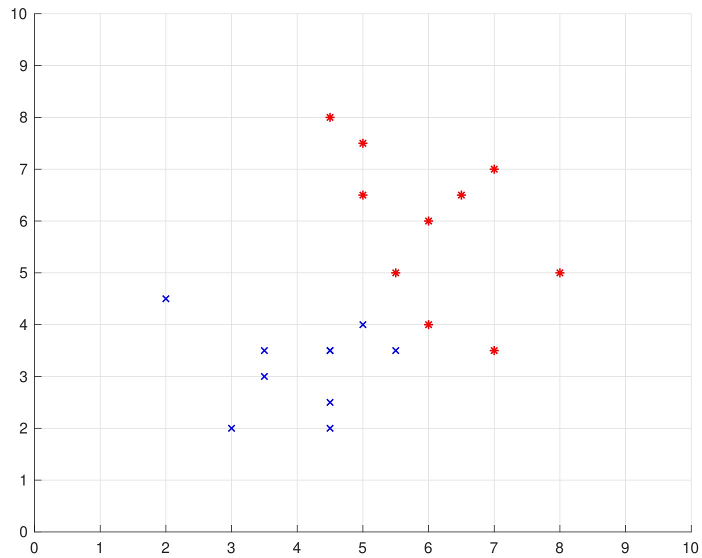
Final Exam - Moed Bet
Total time for the exam: 3 hours!

Important: Please copy the following sentence and sign it: " I am respecting the rules of the exam: Signature:__________ "

1) **Supervised and unsupervised learning (36 pt)**
You are given a training set of 19 samples in Fig 1. 10 red samples marked as red asterisks (*) and the rest are marked as blue Xs i.e., X. Each sample is given by a 2D feature vector. Create a unique test set by using the first 8 digits of your ID number, i.e. for ID number 123456789 the test set is$[(1,2),(3,4),(5,6),(7,8)]$

Feature 1	Feature 2	Label
2	4.5	Blue
3	2	
3.5	3	
3.5	3.5	
4.5	2	
4.5	2.5	
4.5	3.5	
5	4	
5.5	3.5	
4.5	8	Red
5	6.5	
5	7.5	
5.5	5	
6	4	
6	6	
6.5	6.5	
7	3.5	
7	7	
8	5	

Table form



2D Map form

Fig. 1: Training set

Supervised

a) Use the training dataset to classify your test set using KNN for $K = 1, 3$.
b) Fit a single Gaussian to each class
   i) Assume full covariance matrix and draw Gaussians' contour lines and the decision line.
   ii) Assume diagonal covariance matrices and draw the Gaussians' contour lines and decision line.

Unsupervised

Use $(4, 4)$ and $(7, 4)$ as the initial points for the following.

c) Fit a unsupervised GMM, assume diagonal covariance matrices. Draw the Gaussians' contour lines and decision line.
d) Fit a K-means to the data and draw the outcome centroids and decision line.

Ranking the models

e) Rank the supervised models according to the complexity of the inference. Explain your answer.
f) Rank the unsupervised models according to the complexity of the inference. Explain your answer.

2) **MINE (23 points)**
a) **(10 points)** Suppose we are given with a set of i.i.d. samples from two random variables, $\{(x_i, y_i)\}_{i=1}^n$, where $(x_i, y_i,) \sim P_{X,Y}, \quad x_i, y_i \in \mathbb{R}$ and we attempt to estimate $I(X;Y)$. The method is based on MINE $I_n(X,Y)$ as taught in class. Which of the following experiment graphs is a possible description of the estimated MI during training (Figure 2)? Write the indices of the correct graphs in your solution and explain your choice.
b) Now, we wish to estimate the multivariate mutual information between the d-dimensional random vectors, (X^d, Y^d), based on the algorithm and 1-dimensional samples presented in (2a), .
   i) **(8 points)** What criteria should the elements of $X^d, Y^d \in \mathbb{R}^d$ satisfy if we want to use the dateset given in question (2a)?
   ii) **(5 points)** What is the relation between the output of MINE and $I(X^d; Y^d)$ under this criteria and $n \to \infty$?

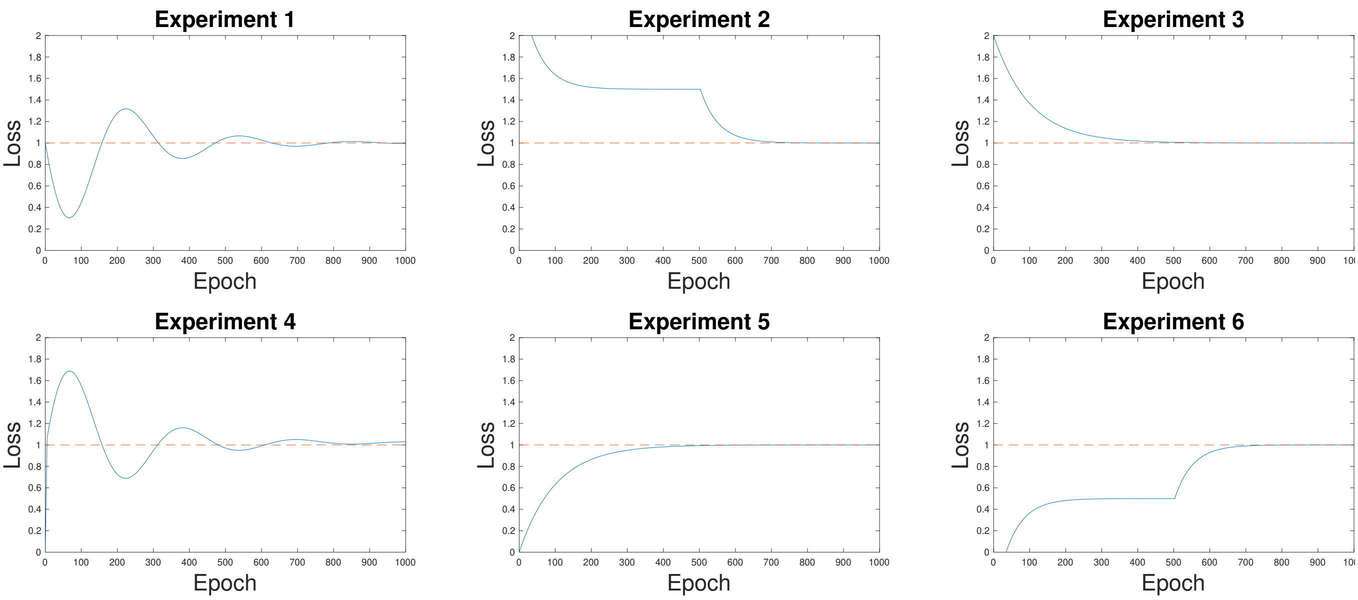


Fig. 2: Experiments - filled line represents the model's loss and dashed line represents the true value.

3) **Bound on the binomial coefficient and types. (51 points)**
Let $k, n \in \mathbb{N}$ such that $k \leq n$. Let $q := \frac{k}{n}$. **Reminder:** The binomial expression is given by

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \tag{1}$$

Furthermore the number of sequences of length n with k ones is $\binom{n}{k}$.
In this question we prove the following inequality:

$$\frac{1}{n+1} 2^{nH_b(q)} \leq \binom{n}{k} \leq 2^{nH_b(q)} \tag{2}$$

where, $H_b(p) = -p \log p - (1-p)\log(1-p)$ is the binary entropy function.
Steps:

a) **(3 points)** Show that:

$$1 = \sum_{i=0}^{n} \binom{n}{i} q^i (1-q)^{n-i} \tag{3}$$

b) **(5 points)** Deduce that for every $i = 0, \ldots, n$:

$$\binom{n}{i} \leq \frac{1}{q^i (1-q)^{n-i}} \tag{4}$$

c) **(5 points)** Using the above inequality show that the right hand side of inequality (2) holds.

d) **(8 points)** Assume the following:

$$\text{argmax}_{i=0,\ldots,n} \binom{n}{i} p^i (1-p)^{n-i} = np, \quad np \in \mathbb{Z} \tag{5}$$

prove the following inequality:

$$\binom{n}{k} \geq \frac{1}{n+1} 2^{nH_b(q)} \tag{6}$$

e) **(8 points)** The type of a Binary sequence of length n with k ones is $\frac{k}{n}$. Provide a lower and upper bound on the number of sequences of length n of a specific type $\frac{k}{n}$.

f) **(3 points)** Is the number of sequences with a fixed type exponential or polynomial?

g) **(8 points)** How many different types there exists if the sequence is Binary and has length n.

h) **(3 points)** Is the number of different types that you calculated in the previous sub-question exponential or polynomial?

i) **(8 points)** For a binary sequence of length n with a type p we actually consider the ratio $\frac{k}{n}$ of such that $\frac{k}{n}$ is the closest number to p. For instance if $p = 0.5$ and $n = 100$ or $n = 101$ then $k = 50$. How many sequences of a fixed type p there exists asymptotically, namely, calculate

$$\lim_{n \to \infty} \frac{1}{n} \log(\text{number of sequences of type } p).$$

Intuition: In this question you have developed an analysis of the number of sequences with a fixed ratio of the alphabet. Note, that you have obtained the entropy expression as a result of a basic combinatorical question. This result is very fundamental and requires only basic knowledge. Hence, we could actually start the course and define the entropy using this fundamental result.

Good Luck!