Introduction to Information and Coding theory 9th Oct 2012
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**Final Exam (Moed Gimel)**

1) **True or False** Please provide a proof. (10 points)
   a) if $X-Y-Z$ form a Markov chain and also $Y-Z-W$ form a Markov chain, then $X-Y-Z-W$ also form a Markov chain
   b) Any Binary source can be reconstructed losslessly after being transmitted through any channel with trinary input and trinary output.
2) **Laplace distribution** (25 points)
   Let $X$ be a continues random variable with $E[X] = 0$ and $E[X^2] = 2\lambda^2$ distributed according to Laplace distribution, i.e.
   $$f_X(x) = \frac{1}{2\lambda} e^{-\frac{|x|}{\lambda}}. \quad (1)$$
   a) Calculate the differential entropy $h(X)$ in nats (logarithm to the base $e$).
      Reminder:
      $$\int f(x) \cdot g'(x)dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)dx. \quad (2)$$
   b) Random variable $Y$ is a result of quantization of $X$ as follows:
      $$Y = Q(x) = \begin{cases} -2, & x < -2 \\ -1.5, & -2 \leq x < -1 \\ -0.5, & -1 \leq x < -0 \\ 0.5, & 0 \leq x < 1 \\ 1.5, & 1 \leq x < 2 \\ 2, & 2 \leq x \end{cases}$$
      Calculate $H(Y)$.
   c) Compress $Y$ using a trinary Huffman code.
   d) What is the average length of the code? What is the ratio with $H(Y)$?
3) **Prefix code for an infinite alphabet** (25 points)
   In the lecture notes and HW we gave the following challenge: find an optimal prefix code for a source with infinite alphabet where the probabilities of the source are $p_1, p_2, p_3, ...$ where $p_i \geq p_j$ if $i < j$, and the entropy of the source is finite.
   A student in the class suggested a procedure that its main idea is to divide the infinite sequence into two parts. The first part is finite and we can apply a Huffman code and the second part is infinite we can apply a Shannon-Fano code. Here are the exact details of the student suggestion:
   - Choose an $N$ and divide the sequence of probabilities into two parts. The first part is $p_1, p_2, ..., p_{N-1}$ and the second part is $p_N, p_{N+1}, p_{N+2}, ....$ Let's denote by $\alpha_N$ the sum of the second part, i.e., $\alpha_N = \sum_{i=N}^{\infty} p_i$.
   - For the first part $p_1, p_2, ..., p_{N-1}$ jointly with $\alpha_N$ namely $p_1, p_2, ..., p_{N-1}, \alpha_N$ apply a Huffman code.
   - Then normalize the probabilities of the second part, i.e., $\frac{p_N}{\alpha_N}, \frac{p_{N+1}}{\alpha_N}, ....$, and find a Shannon-Fano code (which we know that achieves length of $\lceil -\log p(x) \rceil$).
   - For encoding a symbol $i \leq N-1$ use the Huffman Code. For encoding a symboll $i \geq N$ concatenate the codeword corresponding to $\alpha_N$ from the Huffman code with the Shannon-Fano codeword for the corresponding probability $\frac{p_i}{\alpha_N}$.

   Let us denote by $S$ the infimum of the average length over all possible prefix code for the source $p_1, p_2, p_3, ....$ The student claimed the following claims. Please state for each claim if it is True or False and prove or disapprove accordingly.

a) The suggested code is a prefix code.
b) The average length of the prefix code for $p_1, p_2, ..., p_{N-1}, \alpha_N$ is less or equal $S$.
c) The contribution of the Shannon Fano code of $\frac{p_N}{\alpha_N}, \frac{p_{N+1}}{\alpha_N}, ....,$ to the average of the whole code that the student suggested goes to zero as $N \to \infty$ .

4) **Modulo Channel** (25 points)
   a) Consider the DMC defined as follows: Output $Y = X \oplus_2 Z$ where $X$, taking values in $\{0, 1\}$, is the channel input, $\oplus_2$ is the modulo-2 summation operation, and $Z$ is binary channel noise uniform over $\{0, 1\}$ and independent of $X$. What is the capacity of this channel?
   b) Consider the channel of the previous part, but suppose that instead of modulo-2 addition $Y = X \oplus_2 Z$, we perform modulo-3 addition $Y = X \oplus_3 Z$. Now what is the capacity?
   c) Now suppose the noise $Z$ is no longer independent of the input $X$, but is instead described by the following conditional distribution:

$$p(Z = z|X = 0) = \begin{cases} 1/4 \text{ if } z = 0 \\ 3/4 \text{ if } z = 1, \end{cases}$$

   and

$$p(Z = z|X = 1) = 1/2 \quad \text{both for } z = 0 \text{ and } z = 1.$$

   A random code of size $2^{nR}$ is generated uniformly (that is all codewords are drawn i.i.d. $X \sim Bern(0.5)$). Find the value $V$ such that if $R < V$ then the average probability of decoding error (average both across the messages and the randomness in the codebook) vanishes with increasing blocklength while if $R > V$ then it does not. (compute $V$ for both cases, when the channel is mod 2)
   d) Repeat (c) when the channel is mod 3.

5) **Network coding with arbitrary source** (15 points)
   In class/lecture notes we derived the capacity region where we transmit a message with uniform distribution. Sending a message with uniform distribustion is equivalent to analyzing the capacity when the source is Bernouli($\frac{1}{2}$).
   a) (3 points) Explain why sending a message with uniform distribution is equivalent to analyzing the capacity when the source is Bernouli($\frac{1}{2}$).
   b) (4 points) Provide the capacity region of a network coding setting as learned in the class, where there is one source and multiple destination, where the source is distributed i.i.d ∼Bernouli($\alpha$).
   c) (4 points) Provide the achievability proof to the capacity region you gave in (b).
   d) (4 points) Provide the converse proof to the capacity region you gave in (b).

Good Luck!