Mathematical methods in communication (361-2-5891) 17th July 2009
(Dr. Permuter Haim and Mr. Yuval Carmel)

Final Exam

Total time for the exam: 3 hours!

Please sign in and turn the exam sheet with your notebook. A notebook without the test sheet won't be graded.

I am respecting the rules of the exam and won't discuss the exam with anybody till 5pm the 24th of July: Signature____________

1. **True or False** (55 points)

   Let X, Y, Z be discrete random variable. Copy each relation to your notebook and write **true** or **false**. If it's true, prove it. If it is false give a counterexample or prove that the opposite is true.

   For instance:

   - $H(X) \geq H(X|Y)$ is **true**. Proof: In the class we showed that $I(X;Y) > 0$, hence $H(X) - H(X|Y) > 0$.
   - $H(X) + H(Y) \leq H(X,Y)$ is **false**. Actually the opposite is true, i.e., $H(X) + H(Y) \geq H(X,Y)$ since $I(X;Y) = H(X) + H(Y) - H(X,Y) \geq 0$.

   (a) If $H(X|Y) = H(X)$ then X and Y are independent.

   (b) If X, Y, Z form the Markov chain, $X - Y - Z$, then $H(X|Y) \leq H(X|Z)$.

   (c) For any two probability mass functions (pmf) P, Q,

   $$D\left(\frac{P+Q}{2}||Q\right) \leq \frac{1}{2}D(P||Q),$$

   where $D(||)$ is a divergence between two pmfs.

   (d) Let X and Y be two independent random variables. Then

   $$H(X+Y) \geq H(X).$$

(e) $I(X;Y) - I(X;Y|Z) \leq H(Z)$

(f) Let X^n be i.i.d $\sim P_X$. Let A_n and B_n be two sets of sequences X^n such that $\lim_{n\to\infty} \Pr(A_n) = 1$ and $\lim_{n\to\infty} \Pr(B_n) = 1$. For instance, A_n could be the typical set with parameter ϵ, i.e., $A_n = \{x^n : |\frac{1}{n}\log P(x^n) - H(X)| \leq \epsilon\}$. Then, there might be a case where

$$\lim_{n\to\infty} \frac{1}{n} \log |A_n \cap B_n| < H(X),$$

namely the normalized log of the cardinality of $A_n \cap B_n$ could be less than $H(X)$.

(g) There exists a compression software, that whenever we apply it on any file it compresses by 50% the file.

(h) Let R_X be the minimum rate needed to compress an i.i.d. source X without loss. Let $R_{X|Y}$ be the minimum rate needed to compress the same source X without loss where side information Y is known at the encoder and decoder. The source and the side information are i.i.d $(X,Y) \sim P_{X,Y}$. For any alphabet size $\mathcal{X}, \mathcal{Y}$,

$$R_X - R_{X|Y} \leq I(X;Y) + 1$$

(i) There exists $P_{X,Y}$ for which

$$R_X = R_{X|Y}.$$

(j) Let $R(D)$ be a rate distortion function. $R(D)$ is nonincreasing in D. I.e., if $D_1 \geq D_2$ then $R(D_1) \leq R(D_2)$.

(k) Let $R(D)$ be a rate distortion function where the source is memoryless X and the reconstruction is $\hat{X}$. For any distortion we have

$$R(D) \leq \log |\hat{\mathcal{X}}|,$$

where $|\hat{\mathcal{X}}|$ is the cardinality of the alphabet of $\hat{X}$.

2. **Compression** (15 points)

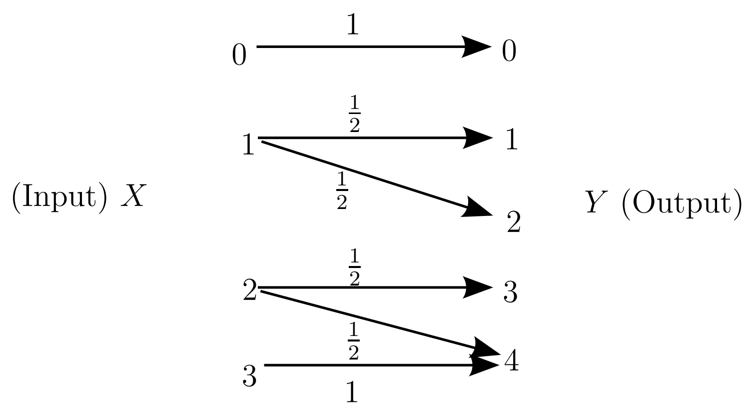
   (a) Give a Huffman encoding into an alphabet of size D = 2 of the following probability mass function:

   $$\left(\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}\right)$$

   (b) Assume you have a file of size 1,000 symbols where the symbols are distributed i.i.d. according to the pmf above. After applying the Huffman code, what would be the pmf of the compressed binary file and what would be the expected length?

3. **Channel capacity** (15 points)

   (a) What is the capacity of the following channel

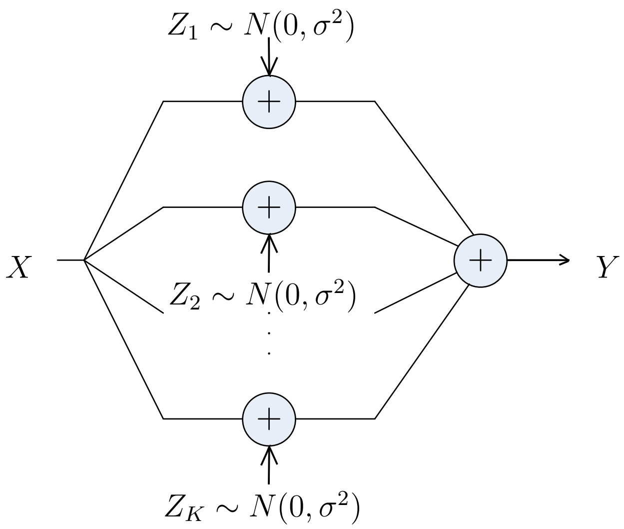


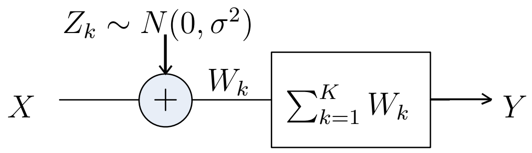
   (b) Provide a simple scheme that can transmit at rate $R = \log_2 3$ bits through this channel.

4. **Diversity** (15 points)

   (a) Calculate the capacity of the following system, where $E[X^2] \leq P$, and the noise sampled $Z_1, ..., Z_K$ are jointly Gaussian with variance σ^2 and independent of each other:

   (b) Compare the result of (a) to the capacity of the following system, with the same power constraint. The output of the channel Y is $\sum_{i=1}^{K} W_k$, where $W_k = X + Z_k$, and K is the number of samples of the noise Z_k. The noise process $Z_1, Z_2, ..., Z_K$ is i.i.d, Gaussian with variance σ^2.





(c) Compare the results in (a) and (b) with simple additive channel where $Z \sim N(0,\sigma^2)$ and $E[X^2] \leq KP$ (like in (a), but with a single branch).