1. (20 points) **Cookies.**

Let

$$V_n = \prod_{i=1}^{n} X_i,$$

where $X_i$ are i.i.d.

$$X_i = \begin{cases} 
1/8, & \text{probability } 1/2, \\
1/2, & \text{probability } 1/2.
\end{cases}$$

Presumably, $X_i$ is the fraction remaining after a single mouse bite.

(a) Let

$$V'_n = \alpha^n.$$

Find the value of $\alpha$ such that $V_n$ and $V'_n$ decrease at the same rate.

For parts (b) and (c), we mix $V_n$ and $V'_n$ as follows. Let

$$Y_i = \lambda \alpha + (1 - \lambda) X_i,$$

where $\lambda \in (0, 1)$. Let

$$V''_n = \prod_{i=1}^{n} Y_i.$$

(b) Is the growth rate of $V''_n$ larger or smaller than log $\alpha$ ?

(c) What is the growth rate of $V''_n$ for $\lambda = 1/2$ ?
2. \textbf{(20 points) Huffman code.}

Find the binary Huffman encoding for

\[ X \sim p = \left( \frac{19}{40}, \frac{8}{40}, \frac{3}{40}, \frac{3}{40}, \frac{2}{40}, \frac{2}{40} \right). \]

3. \textbf{(20 points) Good codes.}

Which of the following codes are possible Huffman codes?

(a) \{0,00,01\}
(b) \{0,10,11\}
(c) \{0,10\}

4. \textbf{(20 points) Errors and erasures.}

Consider a binary symmetric channel (BSC) with crossover probability \( p \).

A helpful genie who knows the locations of all bit flips offers to convert flipped bits into erasures. In other words, the genie can transform the BSC into a binary erasure channel. Would you use his power? Be specific.
5. *(40 points)* **Random walks.**
Consider the following graph with three nodes:

\[
\{X_i\} = \begin{array}{ccc}
1 & - & 2 \\
& - & 3
\end{array}
\]

(a) What is the entropy rate \(H(X)\) of the random walk \(\{X_i\}_{i=1}^\infty\) on this graph?

Now consider a derived process

\[
Y_i = \begin{cases}
0, & \text{if } X_i = 1 \text{ or } 3, \\
1, & \text{if } X_i = 2.
\end{cases}
\]

(b) Is it Markov?
(c) Find the entropy rate \(H(Y)\) of \(\{Y_i\}_{i=1}^\infty\).

Now consider another derived process

\[
Z_i = \begin{cases}
0, & \text{if } X_i = 1 \text{ or } 2, \\
1, & \text{if } X_i = 3.
\end{cases}
\]

(d) Is it Markov?
(e) Find the entropy rate \(H(Z)\) of \(\{Z_i\}_{i=1}^\infty\).

For parts (f), (g), and (h), consider the following graph with three nodes:

\[
\{U_i\} = \begin{array}{ccc}
1 & - & 2 \\
& - & 3
\end{array}
\]

(f) What is the entropy rate \(H(U)\) of the random walk \(\{U_i\}_{i=1}^\infty\) on this graph?

Now consider a derived process

\[
V_i = \begin{cases}
0, & \text{if } U_i = 1 \text{ or } 2, \\
1, & \text{if } U_i = 3.
\end{cases}
\]

(g) Is it Markov?
(h) Find the entropy rate \(H(V)\) of \(\{V_i\}_{i=1}^\infty\).
6. (20 points) **Code constraint.**
What is the capacity of a BSC($p$) under the constraint that each of the codewords has a proportion of 1’s less than or equal to $\alpha$, i.e.,

$$\frac{1}{n} \sum_{i=1}^{n} X_i(w) \leq \alpha, \quad \text{for } w \in \{1, 2, \ldots, 2^{nR}\}.$$  
(Pay attention when $\alpha > 1/2$.)

7. (20 points) **Typicality.**
Let $(X, Y)$ have joint probability mass function $p(x, y)$ given as

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.1</td>
<td>.3</td>
</tr>
<tr>
<td>1</td>
<td>.4</td>
<td>.2</td>
</tr>
</tbody>
</table>

(a) Find $H(X), H(Y),$ and $I(X; Y)$. (Don’t bother to compute the actual numerical values.)

(b) Suppose $\{X_i\}$ is independent and identically distributed (i.i.d.) according to Bern($0.4$), $\{Y_i\}$ is i.i.d. Bern($1/2$), and $X^n$ and $Y^n$ are independent. Find (to first order in the exponent) the probability that $(X^n, Y^n)$ is jointly typical (with respect to the joint distribution $p(x, y)$).

8. (20 points) **Partition.**
Let $(X, Y)$ denote height and weight. Let $[Y]$ be $Y$ rounded off to the nearest pound.

(a) Which is greater $I(X; Y)$ or $I(X; [Y])$?

(b) Why?
9. (20 points) **Amplify and forward.**

We cascade two Gaussian channels by feeding the (scaled) output of the first channel into the second.

\[
\begin{align*}
Z_1 &\sim N(0, N) \\
Z_2 &\sim N(0, N) \\
X_1 &\rightarrow Y_1 \rightarrow X_2 \rightarrow Y_2 \\
P & \rightarrow + \rightarrow \cdot \rightarrow + \\
\end{align*}
\]

Thus noises \( Z_1 \) and \( Z_2 \) are independent and identically distributed according to \( N(0, N) \),

\[
EX_1^2 = EX_2^2 = P,
\]

\[
Y_1 = X_1 + Z_1,
\]

\[
Y_2 = X_2 + Z_2,
\]

and

\[
X_2 = \alpha Y_1,
\]

where the scaling factor \( \alpha \) is chosen to satisfy the power constraint \( EX_2^2 = P \).

(a) (5 points) What scaling factor \( \alpha \) satisfies the power constraint?

(b) (10 points) Find

\[
C = \max_{p(x_1)} I(X_1; Y_2).
\]

(c) (5 points) Is the cascade capacity \( C \) greater or less than \( \frac{1}{2} \log \left(1 + \frac{P}{N}\right) \)?