Homework Set #3
Rates definitions, Channel Coding, Source-Channel coding

1. Rates

(a) **Channels coding Rate:** Assuming you are sending 1024 different messages using 20 usages of a channel. What is the rate (in bits per channel use) that you send.

(b) **Source coding Rate:** Assuming you have a file with $10^6$ Ascii characters, where the alphabet of Ascii characters is 256. After compressing it we get $4\times10^6$ bits. What is the compression rate?

2. Preprocessing the output.
One is given a communication channel with transition probabilities $p(y \mid x)$ and channel capacity $C = \max_{p(x)} I(X; Y)$. A helpful statistician preprocesses the output by forming $\tilde{Y} = g(Y)$, yielding a channel $p(\tilde{y} \mid x)$. He claims that this will strictly improve the capacity.

(a) Show that he is wrong.

(b) Under what conditions does he not strictly decrease the capacity?

3. The Z channel.
The Z-channel has binary input and output alphabets and transition probabilities $p(y \mid x)$ given by the following matrix:

$$Q = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} \quad x, y \in \{0, 1\}$$

Find the capacity of the Z-channel and the maximizing input probability distribution.

4. Using two channels at once.
Consider two discrete memoryless channels $(\mathcal{X}_1, p(y_1 \mid x_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p(y_2 \mid x_2), \mathcal{Y}_2)$ with capacities $C_1$ and $C_2$ respectively. A new channel $(\mathcal{X}_1 \times \mathcal{X}_2, p(y_1 \mid x_1) \times p(y_2 \mid x_2), \mathcal{Y}_1 \times \mathcal{Y}_2)$ is formed in which $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$, are simultaneously sent, resulting in $y_1, y_2$. Find the capacity of this channel.
5. **A channel with two independent looks at Y.**

Let \( Y_1 \) and \( Y_2 \) be conditionally independent and conditionally identically distributed given \( X \). Thus \( p(y_1, y_2|x) = p(y_1|x)p(y_2|x) \).

(a) Show \( I(X; Y_1, Y_2) = 2I(X; Y_1) - I(Y_1; Y_2) \).

(b) Conclude that the capacity of the channel

\[
\begin{array}{c}
X \\
\rightarrow \\
(Y_1, Y_2)
\end{array}
\]

is less than twice the capacity of the channel

\[
\begin{array}{c}
X \\
\rightarrow \\
Y_1
\end{array}
\]

6. **Choice of channels.**

Find the capacity \( C \) of the union of 2 channels \((X_1, p_1(y_1|x_1), Y_1)\) and \((X_2, p_2(y_2|x_2), Y_2)\) where, at each time, one can send a symbol over channel 1 or over channel 2 but not both. Assume the output alphabets are distinct and do not intersect.

(a) Show \( 2^C = 2^{C_1} + 2^{C_2} \).

(b) What is the capacity of this Channel?

7. **Cascaded BSCs.**

Consider the two discrete memoryless channels \((X, p_1(y|x), Y)\) and \((Y, p_2(z|y), Z)\).

Let \( p_1(y|x) \) and \( p_2(z|y) \) be binary symmetric channels with crossover probabilities \( \lambda_1 \) and \( \lambda_2 \) respectively.
(a) What is the capacity $C_1$ of $p_1(y|x)$?
(b) What is the capacity $C_2$ of $p_2(z|y)$?
(c) We now cascade these channels. Thus $p_3(z|x) = \sum_y p_1(y|x)p_2(z|y)$. What is the capacity $C_3$ of $p_3(z|x)$? Show $C_3 \leq \min\{C_1, C_2\}$.
(d) Now let us actively intervene between channels 1 and 2, rather than passively transmitting $y^n$. What is the capacity of channel 1 followed by channel 2 if you are allowed to decode the output $y^n$ of channel 1 and then reencode it as $\tilde{y}^n$ for transmission over channel 2? (Think $W \rightarrow x^n(W) \rightarrow y^n \rightarrow \tilde{y}^n(y^n) \rightarrow z^n \rightarrow \hat{W}$.)
(e) What is the capacity of the cascade in part c) if the receiver can view both $Y$ and $Z$?

8. Channel capacity

(a) What is the capacity of the following channel in Fig. 1 (appears on the next page).
(b) Provide a simple scheme that can transmit at rate $R = \log_2 3$ bits through this channel.
9. **A channel with a switch.** Consider the channel that is depicted in Fig. 2, there are two channels with the conditional probabilities $p(y_1|x)$ and $p(y_2|x)$. These two channels have a common input alphabet $X$ and two disjoint output alphabets $Y_1, Y_2$ (a symbol that appears in $Y_1$ can’t appear in $Y_2$). The position of the switch is determined by R.V $Z$ which is independent of $X$, where $\Pr(Z = 1) = \lambda$.

(a) Show that
\[
i(X; Y) = \lambda I(X; Y_1) + \bar{\lambda} I(X; Y_2). \tag{1}\]

(b) The capacity of this system is given by $C = \max_{p(x)} I(X; Y)$. Show that
\[
C \leq \lambda C_1 + \bar{\lambda} C_2, \tag{2}\]
where $C_i = \max_{p(x)} I(X; Y_i)$.
When is equality achieved?

(c) The sub-channels defined by $p(y_1|x)$ and $p(y_2|x)$ are now given in Fig. 9, where $p = \frac{1}{2}$.
Find the input probability $p(x)$ that maximizes $I(X; Y)$.
For this case, does the equality stand in Eq. (2)? explain!

\begin{align*}
0 & \quad 1-\bar{p} & 0 \\
X & \quad \bar{p} & \quad Y_1 \\
1 & \quad \bar{p} & \quad 1 \\
\end{align*}
\begin{align*}
0 & \quad 1-\bar{p} & 2 \\
X & \quad \bar{p} & \quad Y_2 \\
1 & \quad \bar{p} & \quad 3 \\
\end{align*}

Figure 3: (a) describes channel 1 - BSC with transition probability $p$. (b) describes channel 2 - Z channel with transition probability $p$.

10. **Channel with state**

A discrete memoryless (DM) state dependent channel with state space $\mathcal{S}$ is defined by an input alphabet $\mathcal{X}$, an output alphabet $\mathcal{Y}$ and a set of channel transition matrices $\{p(y|x, s)\}_{s \in \mathcal{S}}$. Namely, for each $s \in \mathcal{S}$ the transmitter sees a different channel. The capacity of such a channel where the state is know causally to both encoder and decoder is given by:

$$C = \max_{p(x|s)} I(X; Y|S).$$  \hfill (3)

Let $|\mathcal{S}| = 3$ and the three different channels (one for each state $s \in \mathcal{S}$) are as illustrated in Fig. 4.

The state process is i.i.d. according to the distribution $p(s)$.

(a) Find an expression for the capacity of the S-channel (the channel the transmitter sees given $S = 1$) as a function of $\epsilon$.

(b) Find an expression for the capacity of the BSC (the channel the transmitter sees given $S = 2$) as a function of $\delta$. 

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(c) Find an expression for the capacity of the Z-channel (the channel the transmitter sees given \( S = 3 \)) as a function of \( \epsilon \).

(d) Find an expression for the capacity of the DM state dependent channel (using formula (3)) for \( p(s) = [\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{6}] \) as a function of \( \epsilon \) and \( \delta \).

(e) Let us define a conditional probability matrix \( P_{X|Y} \) for two random variables \( X \) and \( Y \) with \( |X| = \{0, 1\} \) and \( |Y| = \{1, 2, 3\} \), by:

\[
[P_{X|Y}]_{i=1,j=1}^{3,2} = p(x = j - 1|y = i).
\]  

What is the input conditional probability matrix \( P_{X|S} \) that achieves the capacity you have found in (d)?

11. Modulo Channel

(a) Consider the DMC defined as follows: Output \( Y = X \oplus_2 Z \) where \( X \), taking values in \( \{0, 1\} \), is the channel input, \( \oplus_2 \) is the modulo-2 summation operation, and \( Z \) is binary channel noise uniform over \( \{0, 1\} \) and independent of \( X \). What is the capacity of this channel?

(b) Consider the channel of the previous part, but suppose that instead of modulo-2 addition \( Y = X \oplus_2 Z \), we perform modulo-3 addition \( Y = X \oplus_3 Z \). Now what is the capacity?

12. Cascaded additive modulo-2 channels with dependent noise
Figure 5: Cascaded additive modulo-2 channels.

Consider the additive modulo-2 channel in Fig. 5. The input to the channel, $X$, is binary. The noise $N_1$ is distributed with $\text{Bernoulli}(\epsilon), \epsilon < \frac{1}{2}$. If $N_1 = 1$, the noise $N_2$ is distributed $\text{Bernoulli}(\alpha_1)$. If $N_1 = 0$, the noise $N_2$ is distributed with $\text{Bernoulli}(\alpha_2)$. $N_1$ and $N_2$ are independent of $X$.

User 1 observes the output $Y_1 = X + N_1$. The capacity of the channel between $X$ and $Y_1$ is denoted by $C_1$. User 2 observes the output $Y_2 = X + N_1 + N_2$. The capacity of the channel between $X$ and $Y_2$ is denoted by $C_2$.

(a) Find the capacities $C_1$ and $C_2$ as functions of $\epsilon, \alpha_1, \alpha_2$.

(b) Find the constraints on $\epsilon, \alpha_1, \alpha_2$ such that:
   i. $C_1 = C_2$.
   ii. $C_1 < C_2$
   iii. $C_1 > C_2$

(c) We now introduce a 3rd user which obtains both $(Y_1, Y_2)$. The capacity of the channel between $X$ and $(Y_1, Y_2)$ is $C_3$. Is it true that $C_1 + C_2 \geq C_3$? If not, give a counterexample.