

Cascade and Triangular Source Coding with Side Information at the First Two Nodes

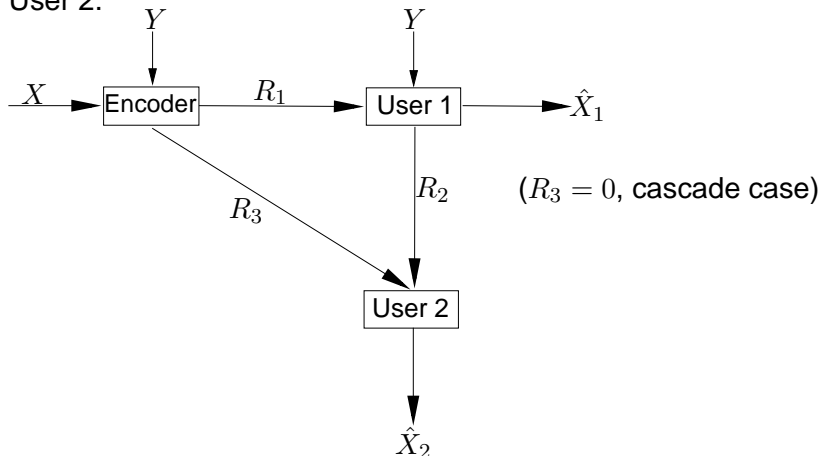
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Jan 2010

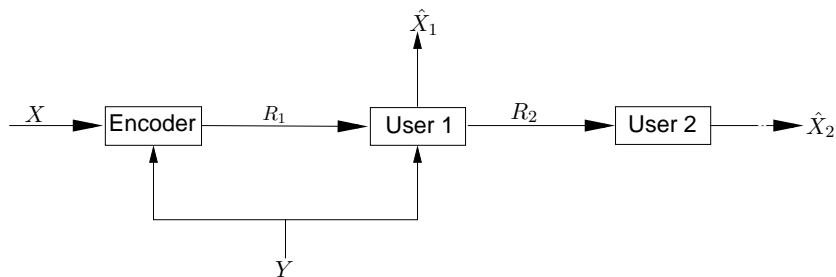
Triangular setting

Side information Y is known to the encoder and User 1, but not to User 2.



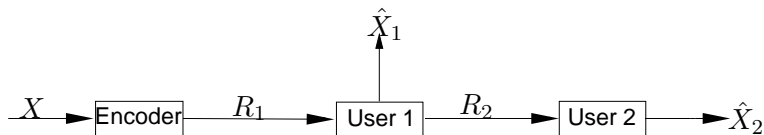
The goal: $\mathbb{E} [d(X^n, \hat{X}_1^n)] \leq D_1, \mathbb{E} [d(X^n, \hat{X}_2^n)] \leq D_2.$

Cascade setting



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Previous work: Cascade case, no side information



Theorem

$$R_1 \geq I(X; \hat{X}_1, \hat{X}_2)$$

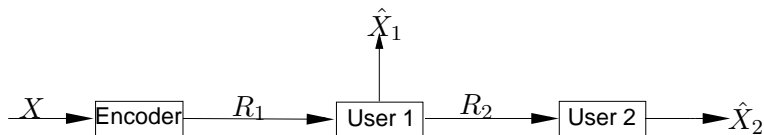
$$R_2 \geq I(X; \hat{X}_2)$$

for some joint distribution $P_{X, \hat{X}_1, \hat{X}_2}$ s.t.

$$\mathbb{E}d_i(X, \hat{X}_i) \leq D_i, \quad i = 1, 2.$$

Yamamoto81

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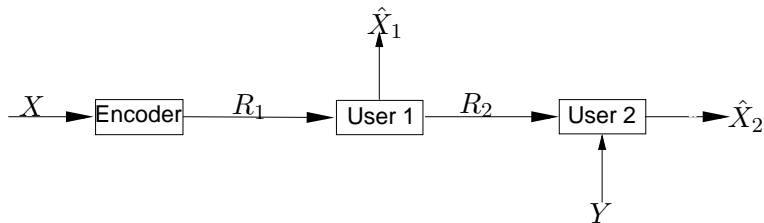
Yamamoto81

Converse: Cut set bound.

Achievability: Use \hat{X}_2 as side information for User 1. Note:

$$R_1 \geq I(X; \hat{X}_1 | \hat{X}_2) + I(X; \hat{X}_2)$$

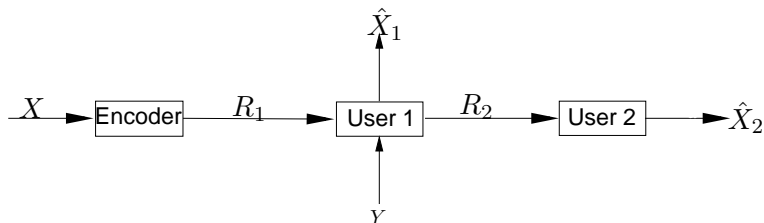
Previous work: Cascade case, with side information at User 2



Vasudevan/Tian/Diggavi06

- provided inner and outer bound
- solved for the Gaussian case
- still open for the general case

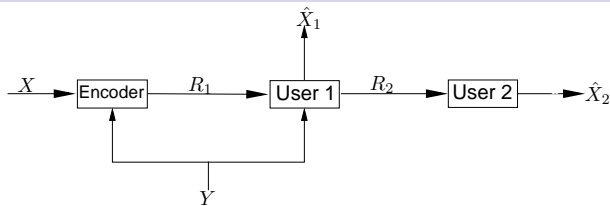
Previous work: Cascade case, with side information at User 1



Su/Cuff/EI Gamal09

- provided inner and outer bound
- both relaying and recompressing information are needed
- still open for the Gaussian and the general case

Cascade with side information



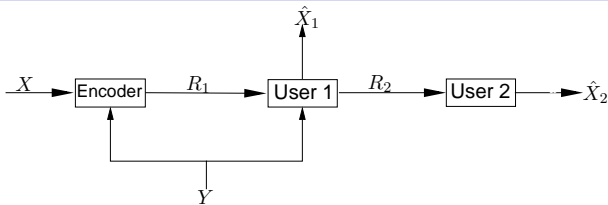
Theorem

$$R_1 \geq I(X; \hat{X}_1, \hat{X}_2 | Y),$$

$$R_2 \geq I(Y, X; \hat{X}_2),$$

for $P_{X,Y,\hat{X}_1,\hat{X}_2}$ s.t. $\mathbb{E}d_i(X, \hat{X}_i) \leq D_i, \quad i = 1, 2.$

Cascade with side information



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Achievability:

$$R_1 \geq \underbrace{I(X, Y; \hat{X}_2) - I(Y; \hat{X}_2)}_{\text{Achieved by binning}} + \underbrace{I(X; \hat{X}_1 | \hat{X}_2, Y)}_{\text{Achieved by Side information}}$$

Converse: Cut set bound.

Cascade Gaussian case

Lemma

It suffices to consider only Gaussian distributions

Cascade Gaussian case

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Proof.

- Given a distribution $P_{X,Y,\hat{X}_1,\hat{X}_2}$ with a covariance K we generate a new distribution $\tilde{P}_{X,Y,\hat{X}_1,\hat{X}_2} \sim \text{Norm}(0, K)$.
- we show

$$\begin{aligned} R_1 &\geq I_P(X; \hat{X}_1, \hat{X}_2 | Y), \\ &\geq I_{\tilde{P}}(X; \hat{X}_1, \hat{X}_2 | Y), \end{aligned}$$

$$\begin{aligned} R_2 &\geq I_P(Y, X; \hat{X}_2) \\ &\geq I_{\tilde{P}}(Y, X; \hat{X}_2 | Y), \end{aligned}$$

- Clearly, the quadratic distortion depends only on the covariance matrix K .

Finding an explicit solution for the Gaussian case

- assume Gaussian distribution and obtain the following optimization problem:

$$\begin{aligned} & \text{maximize (over } \alpha) \quad W \\ & \text{subject to} \quad W \geq \alpha^2 + b\alpha + c \\ & \quad \quad \quad W \leq a'\alpha^2 + b\alpha + c' \end{aligned}$$

a', b, b', c, c' are constants.

- divide into four cases according to the coefficients of the quadratic forms. Each case can be solved analytically.

The explicit rate Gaussian rate-region

W.l.o.g assume $Y = X + Z$

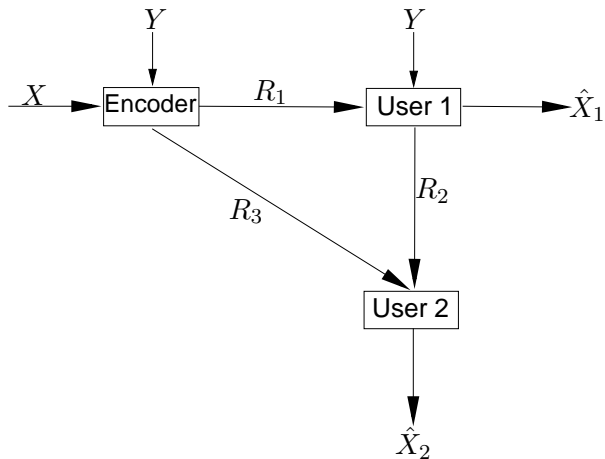
$$R_1(D_1, D_2, R_2) = \frac{1}{2} \max \left(\log \frac{\sigma_{X|Y}^2}{\sigma_{X|W,Y}^2}, \log \frac{\sigma_{X|Y}^2}{D_1}, 0 \right),$$

where $\sigma_{X|W,Y}^2$ is given by the following four cases

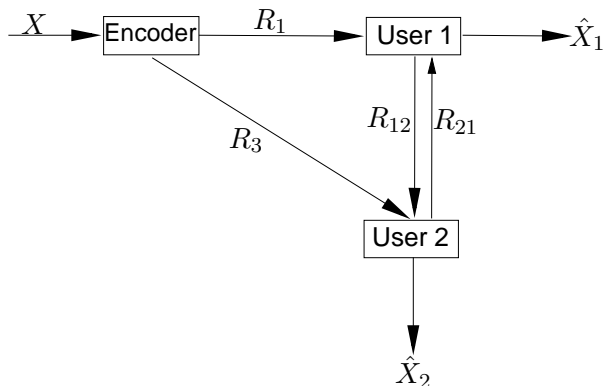
$$\left\{ \begin{array}{ll} \left(\frac{2^{2R_2} D_2 - \sigma_X^2}{\sigma_Z^2 \sigma_X^2 \alpha^2} + \sigma_{X|Y}^{-2} \right)^{-1}, & \text{if } D_2 \leq \sigma_{X|Y}^2 \text{ and } \frac{\sigma_X^2}{D_2} \leq 2^{2R_2} \leq \frac{\sigma_Z^2 (\sigma_X^2 - D_2)}{\sigma_Z^2 \sigma_X^2 - D_2 \sigma_Z^2 - D_2 \sigma_X^2} \frac{\sigma_X^2}{D_2} \\ D_2, & \text{if } D_2 \leq \sigma_{X|Y}^2 \text{ and } 2^{2R_2} \geq \frac{\sigma_Z^2 (\sigma_X^2 - D_2)}{\sigma_Z^2 \sigma_X^2 - D_2 \sigma_Z^2 - D_2 \sigma_X^2} \frac{\sigma_X^2}{D_2} \\ \left(\frac{2^{2R_2} D_2 - \sigma_X^2}{\sigma_Z^2 \sigma_X^2 \alpha^2} + \sigma_{X|Y}^{-2} \right)^{-1}, & \text{if } D_2 \geq \sigma_{X|Y}^2 \text{ and } \frac{\sigma_X^2}{D_2} \leq 2^{2R_2} \leq \frac{\sigma_X^4}{\sigma_X^2 D_2 + \sigma_Z^2 D_2 - \sigma_X^2 \sigma_Z^2} \\ \sigma_{X|Y}^2, & \text{if } D_2 \geq \sigma_{X|Y}^2, \text{ and } 2^{2R_2} \geq \frac{\sigma_X^4}{\sigma_X^2 D_2 + \sigma_Z^2 D_2 - \sigma_X^2 \sigma_Z^2} \end{array} \right.$$

and $\alpha = \left(\frac{\sigma_Z}{\sigma_X} \sqrt{\frac{\sigma_X^2 - D_2}{D_2 - \sigma_X^2 2^{-2R_2}}} - 1 \right)^{-1}$.

Triangular source coding with side information



Triangular source coding **without** side information



Solved by Yamamoto96

Rate-region of triangular source coding with side information

Theorem

$$R_1 \geq I(X; \hat{X}_1, U|Y), \quad (1)$$

$$R_2 \geq I(Y, X; U), \quad (2)$$

$$R_3 \geq I(X; \hat{X}_2|U), \quad (3)$$

for some joint distribution $P(x, y)P(\hat{x}_1, \hat{x}_2, u|x, y)$ for which

$$\mathbb{E}d_i(X, \hat{X}_i) \leq D_i, \quad i = 1, 2. \quad (4)$$

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Achievability:

- show that it does not decrease when $\hat{X}_2 - (X, U) - (\hat{X}_1, Y)$.
- Use cascade achievability and then use U as side information $R_3 \geq I(X; \hat{X}_2|U)$

Gaussian triangular case

Can we claim that it suffices to consider Gaussian distribution (like in the Cascade case)?

- Problem: Let K be a covariance matrix induced by a distribution P . Let \tilde{P} be $\text{Norm}(0, K)$. Then,

$$R_3 \geq I_P(X; \hat{X}_2|U) \not\geq I_{\tilde{P}}(X; \hat{X}_2|U).$$

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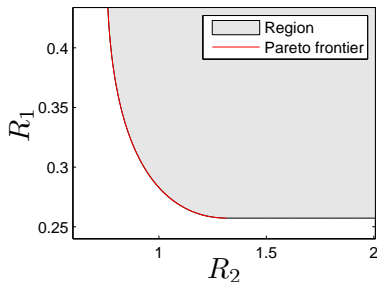
$$R_3 \geq I_P(X; \hat{X}_2|U) \not\geq I_{\tilde{P}}(X; \hat{X}_2|U).$$

- Solution
 - claim that it is enough to consider equalities instead on inequalities using Pareto frontier definition.
 - change the equalities
 - show that Gaussian is optimal for the new set of equations

Pareto frontier

Definition

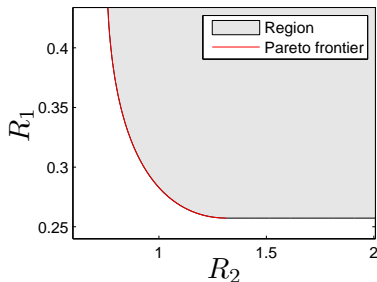
Pareto frontier of a region is the set of all points, for which there is no strictly better point in the region.



Pareto frontier

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Lemma

If two rate-regions, \mathcal{R}_1 and \mathcal{R}_2 have the same Pareto frontier, then they are identical.

Optimality of Gaussian distribution for the Gaussian triangular case

The Pareto frontier is

$$R_1 = I(X; \hat{X}_1, U|Y), \quad (5)$$

$$R_2 = I(Y, X; U), \quad (6)$$

$$R_3 = I(Y, X; \hat{X}_2|U). \quad (7)$$

Equivalently, (7) may be replaced by

$$R_3 + R_2 = I(Y, X; \hat{X}_2, U).$$

Now, we can show that

$$R_1 = I_P(X; \hat{X}_1, U|Y) \geq I_{\tilde{P}}(X; \hat{X}_1, U|Y),$$

$$R_2 = I_P(Y, X; U) \geq I_{\tilde{P}}(Y, X; U),$$

$$R_3 + R_2 = I_P(Y, X; \hat{X}_2, U) \geq I_{\tilde{P}}(Y, X; \hat{X}_2, U).$$

Hence, $\tilde{P} = \text{Norm}(0, K)$ suffices.

Explicit solution for the Gaussian triangular case

- Assume a joint Gaussian distribution $P_{X,Y,U,\hat{X}_1,\hat{X}_2}$
- We note that the third inequality

$$R_3 \geq I(X; \hat{X}_2 | U)$$

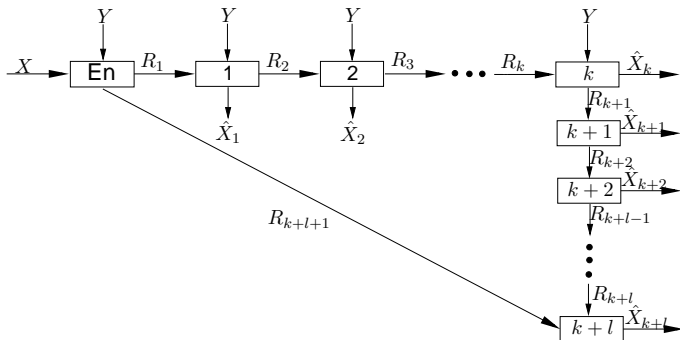
can be always assumed to be equality. Therefore

$$\sigma_{X|\hat{X}_2,U}^2 = \sigma_{X|U}^2 2^{-2R_3}. \quad (8)$$

- Using (8), we obtain an optimization problem identical to the cascade case but D_2 is replaced by $D_2 2^{2R_3}$

$$\mathcal{R}_{triangular}(R_1, R_2, R_3, D_1, D_2) = \mathcal{R}_{cascade}(R_1, R_2, D_1, D_2 2^{2R_3})$$

Extensions to $k + l$ users



Rate-distortion region:

$$R_i \geq I(X; \hat{X}_i, \hat{X}_{i+1}, \dots, \hat{X}_{k+l-1}, U | Y), \quad 1 \leq i \leq k$$

$$R_j \geq I(X; \hat{X}_j, \dots, \hat{X}_{k+l-1}, U), \quad k + 1 \leq j \leq k + l$$

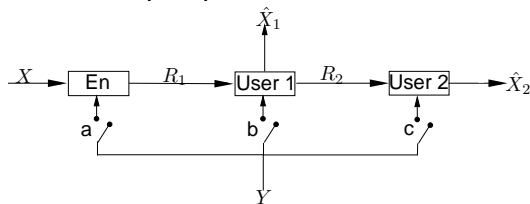
$$R_{k+l+1} \geq I(X; \hat{X}_{k+l} | U),$$

for some distribution

$$P_{X,Y,\hat{X}_1,\hat{X}_2,\dots,\hat{X}_k,U|X,Y} \text{ s.t. } \mathbb{E}d_i(X, \hat{X}_i) \leq D_i, \quad 1 \leq i \leq k + l.$$

Summary

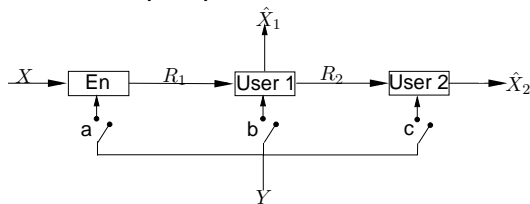
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- Some proof ideas:
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Thank you very much!